NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 6 June 2022 13.30 – 16.30

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.** *Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.*

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 7Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

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Question 1X - Relativity

(i) Consider two coordinate systems S and S' of a 4-dimensional Minkowski spacetime with $x^\mu = (t, x, y, z)$ and $x'^\mu = (u, v, y, z)$. Derive the matrix $\Lambda^\mu{}_\nu$ that relates the two coordinate systems as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ for the coordinate transformation with $u = t - x$, $v = t + x$. [2]

A 4-vector has values $U^\mu = (-1/2, 1/2, 0, 0)$ in S. Calculate its values in S' under this coordinate transformation. [2]

A tensor $F_{\mu\nu}$ is antisymmetric in coordinate system x. Derive the symmetry property of $F_{\mu\nu}$ in coordinate system x'. [2]

Consider the outer product $T^{\mu\nu\rho} = U^\mu \otimes V^\nu \otimes W^\rho$ of three 4-vectors U^μ, V^ν and W^ρ. Show that the contraction $T^{\mu\eta}{}_\eta$ transforms like a 4-vector. [4]

(ii) The Riemann tensor in terms of the metric connection Γ is given by

$$R_{\lambda\mu\nu}{}^\rho = -\frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\lambda} + \frac{\partial \Gamma^\rho_{\lambda\nu}}{\partial x^\mu} + \Gamma^\eta_{\lambda\nu}\,\Gamma^\rho_{\mu\eta} - \Gamma^\eta_{\mu\nu}\,\Gamma^\rho_{\lambda\eta}.$$

Use the Riemann tensor in local Cartesian coordinates to verify that $R_{\lambda\mu\nu\rho} + R_{\lambda\nu\rho\mu} + R_{\lambda\rho\mu\nu} = 0$. [10]

Starting from the twice-contracted Bianchi identity

$$g^{\nu\sigma} g^{\mu\lambda} \left[\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}\right] = 0,$$

show that $\nabla^\mu R_{\rho\mu} = \nabla_\rho R/2$, where $R_{\rho\mu}$ and R are the Ricci tensor and Ricci scalar. [9]

Show that $\nabla^\mu G_{\mu\nu} = 0$, where $G^{\mu\nu}$ is the Einstein tensor. [1]

Question 2Y - Astrophysical Fluid Dynamics

(i) Starting with the equations of fluid dynamics given in the formula sheet, derive the Rankine-Hugoniot relations connecting the pre- and post-shock densities ρ, pressures p, velocities u and temperatures T for the case of a perpendicular shock front in an isothermal fluid. [7]

How are these relations modified if the incoming flow is not perpendicular to the shock front? [3]

(ii) Use the Rankine-Hugoniot relations for an adiabatic gas to show that the Mach number of the pre-shock flow M_1 is related to that of the post-shock flow M_2 by

$$\left[2 + (\gamma - 1)(M_1^2 + M_2^2) - 2\gamma M_1^2 M_2^2\right](M_1^2 - M_2^2) = 0,$$

where γ is the usual ratio of specific heat capacities. [10]

Show that, if the pre-shock flow is supersonic, then the post-shock flow must be subsonic. [6]

As the shock becomes very strong, show that M_2 tends to a constant. Determine this constant. [2]

Evaluate M_2 for a strong shock in a gas which has $\gamma - 1 \ll 1$, and interpret the result physically. [2]

TURN OVER...

Question 3X - Introduction to Cosmology

(i) Consider a Friedmann-Robertson-Walker universe with scale-factor $R(t)$, curvature constant k, containing uniform matter with energy density ρc^2 and pressure p. Show that energy conservation

$$\frac{d\,(\rho R^3)}{dR} = -3\,\frac{p}{c^2}\,R^2,$$

can be written in the form

$$\dot{\rho}\,c^2 = -3\,(\rho\,c^2 + p)\,H,$$

where overdots denote differentiation with respect to time and $H(t) = \dot{R}(t)/R(t)$. [2]

Use the Friedmann equations to show that

$$\dot{H}(t) = -4\pi\,G\,(\rho + p/c^2) + \frac{k\,c^2}{R^2},$$

independent of the cosmological constant Λ. [4]

If $\Lambda = 0$, $p = 0$ and $k > 0$, show that the scale factor reaches a maximum value $R_{\rm max}$ at time $t_{\rm max}$. Show that at $t_{\rm max}$,

$$\dot{H}(t = t_{\rm max}) = -\frac{k\,c^2}{2\,R_{\rm max}^2}.$$

[3]

Give a physical interpretation of this result. [1]

(ii) A Friedmann-Robertson-Walker universe contains non-relativistic matter with density $\rho_{\rm m}$ which decays into a relativistic component with density $\rho_{\rm r}$. The time evolution of the non-relativistic component can be written as

$$\rho_{\rm m}(t) = \frac{A}{R^3}\,{\rm e}^{-\Gamma\,t},$$

where A and Γ are constants and R is the scale-factor. Show that the densities $\rho_{\rm m}$ and $\rho_{\rm r}$ satisfy

$$\begin{aligned} \dot{\rho}_{\rm m} &= -3\,H\,\rho_{\rm m} - \Gamma\,\rho_{\rm m}, \\ \dot{\rho}_{\rm r} &= -4\,H\,\rho_{\rm r} + \Gamma\,\rho_{\rm m}, \end{aligned}$$

where dots denote differentiation with respect to time and $H(t) = \dot{R}(t)/R(t)$. [5]

Show that the density ratio $\theta = \rho_{\mathrm{r}}/\rho_{\mathrm{m}}$ satisfies

$$\dot{\theta}(t) = \Gamma + (\Gamma - H(t))\,\theta(t). \qquad (*)$$

[6]

In a universe dominated by a cosmological constant at late times, $H(t)$ tends to a constant value H_Λ. Show that in this limit, the solution to $(*)$ satisfying the boundary condition $\theta \to 0$ as $t \to 0$ is

$$\theta(t) = \frac{\Gamma}{\Gamma - H_\Lambda}\,\left[\mathrm{e}^{(\Gamma - H_\Lambda)\,t} - 1\right].$$

[6]

Discuss the evolution of $\theta(t)$ for $t \to \infty$ in the two cases $\Gamma > H_\Lambda$ and $\Gamma < H_\Lambda$. [3]

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) Name three scientists credited with key advancements in our understanding of the structure and evolution of stars. For each, explain in a few words the nature of their contribution to the subject. [3]

Explain in a few sentences the main difference between H II regions and planetary nebulae. [2]

Observationally, what differences might you expect in (a) chemical composition, (b) degree of ionisation, and (c) kinematics between H II regions and planetary nebulae? [5]

(ii) Assume that a star evolves at constant mass and that angular momentum is not lost via a wind. How will the rotation speed of the star depend on its radius? [3]

The Sun will ultimately evolve into a white dwarf, with radius $R_{\mathrm{WD}} = 10^7\,\mathrm{m}$. Given that the Sun has a rotation period of 28 days, obtain an estimate of the rotation period of the white dwarf it will eventually become. [5]

Comment on whether this estimate is likely to be an upper or lower limit to the actual value. [2]

Neutron stars are believed to be formed in Type II supernovae as the core of a massive star collapses. If the core had an initial radius $R_{\mathrm{c}} = 10^7\,\mathrm{m}$ and a rotation period of 28 days, estimate the rotation period of the neutron star, assuming $R_{\mathrm{ns}} = 10\,\mathrm{km}$. [4]

Compute the minimum rotation period possible for a neutron star. Compare this to the rotation period of the $R_{\mathrm{ns}} = 10\,\mathrm{km}$ neutron star. [6]

Question 5Z - Statistical Physics

(i) What systems are described by a grand canonical ensemble? If there are N_n particles in microstate n each with energy E_n, write down an expression for the grand canonical partition function, $\mathcal{Z}$, in terms of temperature T, chemical potential μ and Boltzmann constant k_{B}. [4]

Define the grand canonical potential, Φ, in terms of average energy E, temperature T, entropy S, chemical potential μ and the average number of particles $\langle N \rangle$. Write down the relation between Φ and $\mathcal{Z}$. [2]

Using scaling arguments, express $\Phi(T, V, \mu)$ in terms of the pressure, p, and volume, V. [4]

(ii) Consider the grand canonical ensemble as described in Part (i) for a classical ideal gas of non-relativistic particles of mass m in a fixed 3-dimensional volume V. Compute $\mathcal{Z}$ and Φ. [7]

Calculate $\langle N \rangle$, and $\Delta N / \langle N \rangle$, where $(\Delta N)^2 = \langle N^2 \rangle - \langle N \rangle^2$. Comment on the result for $\Delta N / \langle N \rangle$. [4]

Derive the equation of state for the gas. [1]

Using your results and the assumptions of the grand canonical ensemble, derive the equation of state for a classical ideal gas of relativistic particles with mass m and momenta p with energies equal to $\sqrt{p^2 c^2 + m^2 c^4}$. Compute $\Delta N / \langle N \rangle$. [8]

$\Big[$ *You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{-a\,x^2}\,\mathrm{d}x = \sqrt{\pi / a}$ for $a > 0$. $\Big]$

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Let A and $A^\dagger$ respectively be the lowering and raising operator for a one-dimensional quantum harmonic oscillator, with $[A, A^\dagger] = 1$. Also, let $|n\rangle$ be the n^{th} excited state of the oscillator, obeying $N|n\rangle = n|n\rangle$ where $N = A^\dagger A$ is the number operator. Show that $A|n\rangle \propto |n-1\rangle$ and find the constant of proportionality. [5]

For any $z \in \mathbb{C}$, define the *coherent state* $|z\rangle$ as

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \ .$$

Show that $\langle z|z\rangle = 1$ and that $A|z\rangle = z|z\rangle$. [5]

(ii) For the one-dimensional quantum harmonic oscillator described in Part (i), calculate the expectation value $\langle N\rangle$ and uncertainty ΔN of the number operator in the coherent state $|z\rangle$. [8]

Show that the relative uncertainty $\Delta N/\langle N\rangle \to 0$ as $\langle N\rangle \to \infty$. [2]

A harmonic oscillator is prepared to be in state $|z\rangle$ at time $t = 0$. Using your knowledge of the Hamiltonian of the one-dimensional harmonic oscillator, show that the state evolved to time $t > 0$ is still an eigenstate of A and find its eigenvalue. [3]

Calculate the probability that the oscillator is found to be in the original state $|z\rangle$ at time t. [5]

Show that this probability is 1 whenever $t = kT$, where $k \in \mathbb{N}$ and T is the classical period of the oscillator. [2]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) A cluster orbits in a spherically symmetric galaxy whose potential at a radius r is given by

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}, \qquad (*)$$

where M and b are constants.The orbital radius and velocity of the cluster at apocentre are r_a and v_a, where $r_\mathrm{a} v_\mathrm{a} \ll \sqrt{GMb}$ and $r_\mathrm{a} \gg b$. By considering the specific angular momentum of a circular orbit at $r = b$ (or otherwise), show that the orbital radius at pericentre r_p is much smaller than b and that it is approximately given by

$$r_\mathrm{p} \approx r_\mathrm{a} v_\mathrm{a} / \sqrt{GM/b}.$$

[9]

Explain without calculation how this result would change if the effect of gravitational drag was included. [1]

(ii) Show that the path of the orbit for a particle with specific angular momentum h in a spherical potential with radial acceleration f_r is given by

$$\frac{d^2u}{d\phi^2} + u = -\frac{f_r}{h^2 u^2},$$

where $u = r^{-1}$ and ϕ is the azimuthal angle in the orbital plane. [7]

A cluster orbits at large radii ($r \gg b$) in a galaxy whose potential is given by $(*)$ from Part (i). By neglecting terms of order $(b/r)^2$ or higher, derive an approximation to f_r and to the path of the cluster's orbit $r(\phi)$. [8]

Describe the orbit and determine the rate at which it undergoes apsidal precession. [5]

TURN OVER...

Question 8Y - Topics in Astrophysics

(i) Applying the instantaneous recycling assumption to a simple closed box model of galactic chemical evolution, the evolution of the mass of metals is given by

$$\frac{\mathrm{d}(ZM_\mathrm{g})}{\mathrm{d}t} = -Z\,\psi + Z\,\psi\,R + y_z\,(1-R)\,\psi, \qquad (*)$$

where Z is the mass fraction of metals in the gas, M_g is the mass of gas in the closed box, ψ is the star formation rate ($M_\odot\,\mathrm{yr}^{-1}$), R is the mass fraction of stellar material returned to the interstellar medium by each generation of stars, and y_z is the yield of new metals per stellar generation. Explain the physical meaning of each of the three terms on the right hand side of $(*)$, and give the corresponding expression for the evolution of the gas mass $\mathrm{d}M_\mathrm{g}/\mathrm{d}t$. [3]

Provide integral expressions for R and y_z in terms of stellar mass m, the mass fraction of newly produced metals $p_{z,m}$, the mass of stellar remnants $M_\mathrm{rem}(m)$, and the stellar initial mass function $\phi(m)$. [3]

Show that in the case of this simple model where the system starts with pure primordial gas, Z is given by

$$Z = y_z \ln\left(1/\mu\right),$$

where $\mu = M_\mathrm{g}/M_\mathrm{t}$, and M_t is the total mass of the system. [4]

(ii) An important extension to the simple closed box model for galactic chemical evolution presented in Part (i) is to consider outflow of gas. In this leaky box model, outflow occurs at a rate given by

$$W = \lambda\,(1-R)\psi,$$

where $\lambda \geq 0$ is a parameter describing the efficiency of the outflowing wind and other variables have the same meaning as in Part (i). For the case of this leaky box model, derive an expression for the metallicity of the gas as a function of λ, μ, and y_z, where μ and y_z are as defined in Part (i). [10]

What is the effect of considering outflows on the rate of metal enrichment of the interstellar gas? [2]

The occurrence rate of Jupiter-mass planets, f_J, is described by

$$f_\mathrm{J} = 0.1\,m\,Z_*/Z_\odot,$$

where Z_* is the stellar metallicity (the mass fraction of metals in the star), $Z_\odot = 0.01$ is the solar metallicity, and m is the stellar mass in units of solar mass. A survey of stars has found an occurrence rate of 3% for Jupiter-mass planets around stars of 1–2 $M_\odot$. What is the metallicity of the stars required to explain this observation? You may assume an initial mass function of the form $\phi(m)\mathrm{d}m \propto m^{-5/2}\mathrm{d}m$. [5]

Comment on the likely age of the stars relative to the age of the Sun. [1]

Giving your assumptions, estimate the likely bulk metallicity of the planets, assuming they formed by core accretion. [2]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 7 June 2022 13.30 – 16.30

ASTROPHYSICS - PAPER 2

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Question 1X - Relativity

(i) In Minkowski space, an inertial frame S' is moving along the x-axis of an inertial frame S with coordinates (t, x, y, z). An observer at rest in S' sees a flash of red light when they are at x_{R} and a flash of blue light when they are at x_{B}. The red flash happens before the blue and the time difference between the two events measured by the observer at rest in S' is $\Delta t'_{\mathrm{RB}}$. Find the velocity of S' relative to S. [3]

Comment on the order of the two flashes and the time elapsed between the red and the blue flash for an observer at rest in S. [2]

A muon with a total energy of 2 GeV and lifetime $\tau = 2.2\,\mu$s is created in the Earth's atmosphere at an altitude $L = 10\,$km above the ground. The muon moves vertically towards Earth. Does the muon reach the ground? [5]

[The muon rest mass is $m_\mu = 105.7$ MeV$/c^2$.]

(ii) A particle of mass m, speed u, energy E and momentum p moves in the positive x-direction of an inertial frame in Minkowski space. Using the 4-momentum or otherwise, verify that $E^2 - p^2c^2$ is invariant under a standard Lorentz boost in the x-direction. [5]

Calculate the value of this Lorentz invariant. [2]

Show, in detail, that an electron-positron pair cannot be created from a single isolated photon and comment on how pair creation therefore normally proceeds. [7]

A pion with rest mass m_π decays into a muon with rest mass m_μ and a massless neutrino. Calculate the kinetic energy of the muon in the rest frame of the pion in terms of m_π and m_μ. [6]

Question 2Y - Astrophysical Fluid Dynamics

(i) A fully ionized hydrogen plasma can be considered as the superposition of a proton fluid (number density n^+ and bulk velocity $\mathbf{u}^+$) and an electron fluid (number density n^- and bulk velocity $\mathbf{u}^-$). Starting from the continuity equation for each fluid, derive the standard form for charge conservation

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

As part of your answer you should explicitly relate the net charge density q and the net current density $\mathbf{j}$ to the quantities given above. [5]

Further, this plasma has no externally imposed magnetic field and is "cold". By considering the corresponding equation of motion for the charged fluids in the centre-of-mass rest frame of the plasma, i.e.,

$$(m^\pm)\,\frac{\partial \mathbf{u}^\pm}{\partial t} = \pm\, e\,\mathbf{E},$$

where $m^+ = m_{\mathrm{p}}$ and $m^- = m_{\mathrm{e}}$, show that any small charge imbalances oscillate with angular frequency $\omega_{\mathrm{pl}} \approx \sqrt{e^2\, n^- / (m_{\mathrm{e}}\, \epsilon_0)}$. [5]

(ii) Suppose that fluctuations within a plasma have a spatial scale ℓ and timescale τ. Use Maxwell's equations to show that

$$\left|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right| \Big/ |\nabla \times \mathbf{B}| \sim \frac{u^2}{c^2},$$

where $u \sim \ell/\tau$ is a characteristic velocity. [5]

Use this result to justify the neglect of displacement current in standard non-relativistic magnetohydrodynamics (MHD). [3]

Consider a plasma in a state of equilibrium at rest ($\mathbf{u} = 0$), with uniform density $\rho = \rho_0$, uniform pressure $p = p_0$, and threaded by a uniform vertical magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Suppose that it is subject to perturbations with plane wave form, $\exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$, that do not necessarily satisfy the condition $\ell/\tau \ll c$. In this case the standard momentum equation of MHD is replaced by

$$\rho\left(\frac{\partial u}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \frac{1}{\mu_0}\left(\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B},$$

where $\mathbf{E}$ is the electric field induced by the magnetic field perturbations.

CONTINUE OVER...

Restricting attention to perturbations that propagate in the $\hat{\mathbf{z}}$-direction and are transverse (that is velocity perturbations orthogonal to $\mathbf{k}$), show that

$$\omega^2 \left(\rho_0 + \frac{B_0^2}{\mu_0 c^2} \right) = \frac{B_0^2}{\mu_0} k^2.$$

[10]

Derive an expression for the phase velocity of these waves and comment on the cases of (a) a very weak, and (b) a very strong magnetic field. [2]

Question 3X - Introduction to Cosmology

(i) The cosmic microwave background (CMB) radiation is described accurately by a blackbody distribution. The proper number density of photons with frequency in the range ν and $\nu + \mathrm{d}\nu$ is given by

$$dn_\gamma(\nu) = \frac{8\pi}{c^3} \frac{\nu^2\,\mathrm{d}\nu}{[\exp\left(h_\mathrm{P}\nu/(k_\mathrm{B}T)\right) - 1]},$$

where k_B and h_P are the Boltzmann and Planck constants. The present day temperature is, $T_0 = 2.726$ K. Show that the number density of CMB photons is

$$n_\gamma \approx 60.42 \left(\frac{k_\mathrm{B}T}{h_\mathrm{P}c}\right)^3. \qquad (*)$$

Evaluate n_γ at the present day. [5]

Calculate the baryon-to-photon ratio $\eta = n_\mathrm{b}/n_\gamma$ in terms of the parameter combination $\Omega_\mathrm{b}h^2$, where Ω_b is the density of baryons in units of the critical density $\rho_\mathrm{c} = 1.88 \times 10^{-26}\,h^2\,\mathrm{kg\,m^{-3}}$, and h is the Hubble constant H_0 in units of $100\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. [4]

How does η evolve with time? [1]

[You may assume: $\int_0^\infty x^2(\mathrm{e}^x - 1)^{-1}\,dx = 2.4041$ and $\int_0^\infty x^2\mathrm{e}^{-x^2}\,dx = \sqrt{\pi}/4$.]

(ii) For a uniform distribution of particles of species i and rest mass m_i in thermal equilibrium at temperature T, the number density of particles with momentum between p and $p + dp$ is given by

$$\mathrm{d}n_i(p) = \frac{4\pi}{h_\mathrm{P}^3}\,g_i\,\frac{p^2\mathrm{d}p}{[\exp\left((E-\mu_i)/(k_\mathrm{B}T)\right) \pm 1]}, \qquad \begin{cases} +1 \text{ for fermions,} \\ -1 \text{ for bosons,} \end{cases}$$

where $E^2 = p^2c^2 + m_i^2c^4$, g_i is the number of spin states, μ_i is the chemical potential and k_B and h_P are the Boltzmann and Planck constants. In the non-relativistic limit, show that the number density of particles is

$$n_i \approx \frac{\pi^{3/2}}{h_\mathrm{P}^3}\,g_i\,(2m_ik_\mathrm{B}T)^{3/2}\exp\left(-(m_ic^2 - \mu_i)/(k_\mathrm{B}T)\right).$$

[6]

CONTINUE OVER...

Prior to recombination, the reaction

$$\mathrm{H} + \gamma \longleftrightarrow \mathrm{p} + \mathrm{e}^-,$$

is in statistical equilibrium. Assuming that the particles remain in statistical equilibrium, show that the ionization fraction $x = n_\mathrm{p}/(n_\mathrm{p} + n_\mathrm{H})$ satisfies

$$\frac{1-x}{x} \approx n_\mathrm{p} \left(\frac{h_\mathrm{P}^2}{2\pi m_e k_\mathrm{B} T} \right)^{3/2} \exp\left(Q/(k_\mathrm{B}T)\right), \qquad (**)$$

where n_p, n_H, m_p and m_H are the number densities and masses of protons and hydrogen atoms, m_e is the electron mass, and $Q = (m_\mathrm{p} + m_\mathrm{e} - m_\mathrm{H})\, c^2$ is the binding energy of hydrogen.

Using the result $(*)$ of Part(i), show that $(**)$ can be written as

$$\frac{1-x}{x^2} \approx 3.84\, \eta \left(\frac{k_\mathrm{B}T}{m_e c^2} \right)^{3/2} \exp\left(Q/(k_\mathrm{B}T)\right),$$

where η is the baryon-to-photon ratio $\eta = n_\mathrm{b}/n_\gamma$. [8]

Why does recombination occur at temperatures with $k_\mathrm{B}T \ll Q$? [4]

Why does $(**)$ give a poor description of recombination? [2]

Question 4Z - Structure and Evolution of Stars

(i) A star, with an apparent magnitude $m_{\rm V} = 2.5$, is found to have a parallax of 0.002 arcsecond. What is its absolute magnitude in V? What type of star would have such a value of $M_{\rm V}$, knowing that $M_{\rm V} = +0.6$ for a star of spectral type A0V? [3]

The star explodes as a Type II supernova, increasing its luminosity by a factor of 50,000. Calculate the new values of $m_{\rm V}$ and $M_{\rm V}$. [1]

Spectroscopy of the supernova remnant reveals P-Cygni profiles for the Balmer lines of hydrogen. The shortest wavelength of the Hα profile is measured to be $\lambda_{\rm min} \simeq 6345$ Å. The rest-frame wavelength of Hα is 6563 Å. Deduce the expansion velocity of the remnant. [2]

If this expansion velocity is maintained for 100 days, would it be possible to resolve the supernova remnant from images obtained with a ground-based telescope? [4]

(ii) A gas cloud collapses to form a star cluster. The number of stars formed with masses in the range from M to $M + {\rm d}M$ is given by,

$${\rm d}N = \phi(M)\,{\rm d}M.$$

The function $\phi(M)$ is observed to have the form, $\phi(M) \propto M^{-2.35}$. Assuming that the total luminosity of the cluster is dominated by stars on the main sequence for which $L \propto M^{3.5}$, calculate the change in the luminosity of the cluster as the cluster ages and the main sequence turn-off moves from $2\,M_\odot$ to $1\,M_\odot$. [5]

The function $\xi(M)$ denotes the amount of mass contained in stars with masses in the range from M to $M + dM$,

$$M\,{\rm d}N = \xi(M)\,{\rm d}M.$$

Deduce a functional form for $\xi(M)$. [1]

Let $M_{\rm min}$ and $M_{\rm max}$ respectively denote the minimum and maximum mass of stars on the main sequence. As stars leave the main sequence, those with mass $M \geq M_{\rm SN}$ return a fraction of their mass equal to $M_{\rm exp} = 0.9\,M$ to the interstellar medium in a supernova explosion. Stars with $M < M_{\rm SN}$ end their lives as a white dwarf with mass $M_{\rm WD} = 0.6\,{\rm M}_\odot$, returning the remainder of their main sequence mass to the interstellar medium. Derive an expression for the fraction of the total mass of the cluster returned to the interstellar medium when the main sequence turn-off occurs at a mass of $1\,{\rm M}_\odot$. [9]

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By choosing suitable values of $M_{\rm min}$, $M_{\rm max}$, and $M_{\rm SN}$, provide a numerical estimate of the fraction of the cluster mass returned to the interstellar medium. [2]

Explain what is meant by a 'top-heavy' IMF, and put forward an observational test that may indicate that the IMF of a burst of star formation is top-heavy. [3]

Question 5Z - Statistical Physics

(i) What systems are described by microcanonical ensembles and canonical ensembles? [2]

Consider the Gibbs formula for entropy, $S = -k_\mathrm{B} \sum_n p(n) \ln p(n)$, where $p(n)$ is the probability of being in microstate n and k_B is the Boltzmann constant. Show how maximising entropy subject to appropriate constraints leads to the correct forms of the probability distributions for (a) the microcanonical ensemble and (b) the canonical ensemble. [8]

(ii) Derive an expression for the entropy in the canonical ensemble in terms of the partition function Z and temperature T. [5]

A system consists of N non-interacting particles fixed at points in a lattice in thermal contact with a reservoir at temperature T. Each particle has three states with energies $-\epsilon, 0, \epsilon$, where $\epsilon > 0$ is a constant. Compute the average energy, E, and the entropy, S. [8]

Evaluate E and S in the limits $T \to \infty$ and $T \to 0$. [4]

Describe a configuration for this system that would have negative temperature. Justify your answer. [3]

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a basis of S_z eigenstates for a spin-$\frac{1}{2}$ particle. Show that the eigenvalues are $+\hbar/2$ and $-\hbar/2$. . [5]

Derive from the eigenvalue equation that the respective, normalised eigenstates $|\uparrow_\theta\rangle$ and $|\downarrow_\theta\rangle$ of $\mathbf{n}\cdot\mathbf{S}$, where $\mathbf{n} = (\sin\theta, 0, \cos\theta)$, are

$$|\uparrow_\theta\rangle = \cos(\theta/2)\ |\uparrow\rangle + \sin(\theta/2)\ |\downarrow\rangle, \qquad |\downarrow_\theta\rangle = -\sin(\theta/2)\ |\uparrow\rangle + \cos(\theta/2)\ |\downarrow\rangle.$$

[5]

[*You may assume that* $\tan(\theta/2) = [1-\cos(\theta)]/\sin(\theta) = \sin(\theta)/[1+\cos(\theta)]$. *The Pauli sigma matrices are given by*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

]

(ii) Two spin-$\frac{1}{2}$ particles, with eigenstates and eigenvalues as described in Part (i), are in the combined spin state

$$|\psi\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}.$$

Show that this state is unchanged under the substitution

$$(|\uparrow\rangle, |\downarrow\rangle) \mapsto (|\uparrow_\theta\rangle, |\downarrow_\theta\rangle).$$ [4]

Hence, show that $|\psi\rangle$ is an eigenstate, with eigenvalue zero, of each Cartesian component of the combined spin operator $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$, where $\mathbf{S}^{(i)}$ is the spin operator of the i^{th} particle. [8]

Two spin-$\frac{1}{2}$ particles are in the spin state

$$|\chi\rangle = \frac{|\uparrow\rangle|\downarrow_\theta\rangle - |\downarrow\rangle|\uparrow_\theta\rangle}{\sqrt{2}}.$$

A measurement of S_z for the first particle is carried out, followed by a measurement of S_z for the second particle. List the possible outcomes for this pair of measurements and find the total probability, in terms of θ, for each pair of outcomes to occur. [4]

For which of these outcomes is the system left in an eigenstate of the combined total spin operator $\mathbf{S}\cdot\mathbf{S}$, and what are the corresponding eigenvalues?

[4]

[*The Pauli sigma matrices are as given in Part (i).*]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Consider a spherically symmetric galaxy with a Hernquist density profile, in which the density at radius r given by

$$\rho(r) = \frac{M}{2\pi}\frac{a}{r}\frac{1}{(a+r)^3}, \qquad (*)$$

where M and a are constants. Show that the corresponding potential is

$$\Phi(r) = -\frac{GM}{(a+r)}.$$

[3]

Find an expression for the total mass contained within radius r and compare this value, in the limits $r \to 0$ and $r \to \infty$, to the corresponding values when $a = 0$. [5]

Explain whether the Hernquist model is a good description of the potential in the outer parts of disc galaxies. [2]

(ii) The gravitational drag force on a body of mass m moving at velocity $\mathbf{v}$ through a medium of density ρ can be written in the form

$$\mathbf{F}_\mathrm{D} \sim -\pi\rho\frac{G^2m^2}{v^3}\mathbf{v},$$

where $v = |\mathbf{v}|$. Explain what is meant by the gravitational drag force. [3]

Assuming that the impact parameter at which particles experience a deflection by $90°$ is $b = Gm/v^2$, explain why the expression for the drag is of this form. [5]

A dwarf galaxy of mass m is moving radially inwards through a larger galaxy of mass M, whose radial density profile is described by the Hernquist profile given by $(*)$ in Part (i). Assuming that the dwarf galaxy arrives in the outer parts of the galaxy with its free-fall velocity from infinity, estimate the radius, r_drag, at which the gravitational drag force becomes the dominant component of the total force experienced by the dwarf galaxy. [10]

Evaluate r_drag in the case $M = 10^{11}M_\odot$, $m = 10^8 M_\odot$, $a = 1\,\mathrm{kpc}$. [1]

Explain how the velocity of the dwarf galaxy evolves at orbital radii inside r_drag. [1]

TURN OVER...

Question 8Y - Topics in Astrophysics

(i) Describe the possible outcomes of the collision between two planetesimals in terms of the specific energy of impact, Q, and the size of the objects. [6]

Estimate the minimum radius a planetesimal would need to have to survive impact from an interstellar asteroid travelling with a relative velocity of $90\,\mathrm{km\,s^{-1}}$ and having a mass of 90,000 kg. You may assume a density of $2000\,\mathrm{kg\,m^{-3}}$ for the planetesimal. [4]

(ii) A spherical planetesimal of radius 200 km is slowly drifting towards the Sun. The planetesimal completes a full rotation once every 3 hours, and its axis of rotation is perpendicular to the plane of its orbit around the Sun. It has a density of $2000\,\mathrm{kg\,m^{-3}}$ and a material strength of 1 MPa. Determine the distance from the Sun where the planetesimal would be tidally disrupted. You may assume that the deformation and relaxation of the planetesimal occur on much shorter timescales than its rotation and that the equation for the tidal force F_{t} per unit mass experienced by the planetesimal is given by

$$\frac{F_{\mathrm{t}}}{M_{\mathrm{p}}} = 2\frac{GM_{\odot}R_{\mathrm{p}}}{r^3},$$

where M_{p} and R_{p} are the planetesimal mass and radius, and r its distance from the Sun. [3]

At the tidal disruption location calculated above, estimate the proportion of energy input to the planetesimal from tidal heating relative to that from solar radiation. [9]

Estimate how close the planetesimal can approach the Sun before tidal heating alone causes it to melt (that is ignoring energy input from solar radiation). You may assume a melting temperature of 1400 K for the planetesimal and rapid heat transport through the body. [6]

As the planetesimal heats up, what feedback occurs to diminish the magnitude of tidal heating, limiting the temperature rise and degree of melting? [2]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 9 June 2022 9am– 12pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.** *Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.*

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 7Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X - Relativity

(i) The energy-momentum tensor of an ideal fluid filling a universe has the form $T_{\mu\nu} = \alpha\, u_\mu\, u_\nu + \beta\, g_{\mu\nu}$, where u_μ is the four-velocity of the fluid and $g_{\mu\nu}$ is the metric. Derive α and β in terms of the mass density ρ and the pressure p. [4]

The Einstein field equations with a cosmological constant Λ can be written as

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = -\frac{8\pi\, G}{c^4}\, T_{\mu\nu},$$

where $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar, respectively. Consider a universe with a positive cosmological constant and a vanishing energy-momentum tensor. Derive the equation of state $p(\rho)$ of an ideal fluid that gives equivalent field equations with a vanishing cosmological constant. Give a possible physical interpretation of such a fluid. [4]

Calculate the trace of $T_{\mu\nu}$ for such a fluid. [2]

(ii) In an expanding universe described by the Robertson-Walker metric, the line element is given by

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - a^2(t) \left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega^2 \right] = c^2 \mathrm{d}t^2 - a^2(t)\, \gamma_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j,$$

where $x^\mu = (t, r, \theta, \phi)$, K is the curvature constant, and $a(t)$ the scale factor. The Ricci tensor in terms of the metric connection Γ is given by

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\, \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\, \Gamma^\lambda_{\lambda\eta}.$$

Derive that

$$R_{00} = 3\, \frac{\ddot{a}}{a},$$

where overdots denote differentiation with respect to time t. [8]

The remaining values of the Ricci tensor are $R_{0i} = 0$ and

$$R_{ij} = -\frac{1}{c^2} \left(\ddot{a} a + 2\dot{a}^2 + 2Kc^2 \right) \gamma_{ij}.$$

Use the Einstein field equations as given in Part (i) to derive the Friedmann equations for an empty universe with $T_{\mu\nu} = 0$ and a non-vanishing cosmological constant Λ. [6]

Derive the minimum scale factor of such a universe in terms of Λ and K for the case where the universe is closed and the cosmological constant is positive. [4]

Calculate the expansion history $a(t)$ of a flat and empty universe with a positive cosmological constant and comment on the fate of such a universe. [2]

TURN OVER...

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a galaxy cluster whose gravitational potential is dominated by a spherically symmetric dark matter halo and is given by

$$\Phi = -a^2 \ln \left[1 + \left(\frac{r}{r_0} \right)^2 \right],$$

where a is a positive constant, r is the distance from the cluster centre, and $r_0 = 20\,\mathrm{kpc}$. An isothermal intracluster medium (ICM) with isothermal sound speed c_s is in a state of hydrostatic equilibrium within this potential, where $c_\mathrm{s} \approx a$. Derive that the density profile for the ICM is

$$\rho = \rho_0 / [1 + (r/r_0)^2],$$

where ρ_0 is a constant. [5]

If the ICM has temperature $T = 7 \times 10^7\,\mathrm{K}$, and a central electron number density $n_\mathrm{e} = 5 \times 10^{-2}\,\mathrm{cm}^{-3}$, calculate the radius at which the electron mean-free-path is 20 kpc. [5]

[*You may assume that the electron mean-free-path is given by*

$$\lambda_\mathrm{e} = 2.3 \left(\frac{T}{10^7\,\mathrm{K}} \right)^2 \left(\frac{n_\mathrm{e}}{10^{-2}\,\mathrm{cm}^{-3}} \right)^{-1} \mathrm{pc},$$

where n_e is the electron number density.]

(ii) An atmosphere is in hydrostatic equilibrium with a gravitational field $\mathbf{g} = -g(z)\,\hat{\mathbf{z}}$. This atmosphere can support *internal gravity waves* with wavevector $\mathbf{k} = (k_x, k_y, k_z)$ and a dispersion relation

$$\omega^2 = \frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} N^2, \qquad (*)$$

where N is the Brunt-Väisälä frequency given by

$$N^2 \equiv \frac{g(z)}{\gamma} \frac{\partial}{\partial z} \ln \left(P \rho^{-\gamma} \right).$$

Explain the connection between this dispersion relation and the Schwarzschild criterion for convective stability. [3]

Show that the wavevector and group velocity ($\mathbf{v}_\mathrm{g} = \nabla_\mathbf{k}\, \omega$) of these waves are orthogonal to each other. [6]

Assuming that the dispersion relation $(*)$ is a valid local approximation for the ICM of Part (i), sketch N^2 as a function of radius, noting specifically the behaviour for $r \ll r_0$ and $r \gg r_0$. [7]

The orbital motions of galaxies at $r \sim r_0$ excite internal gravity waves in the ICM with frequency $\omega_0 < N$. With reference to your sketch of N^2, explain why oscillatory modes will be confined to a region $r_- < r < r_+$, and give an implicit equation for $r_\pm$. [4]

TURN OVER...

Question 3X - Introduction to Cosmology

(i) The energy-momentum tensor of a uniform perfect fluid with energy density ρ and pressure p at rest in locally Minkowski coordinates has components

$$T_{00} = \rho, \quad T_{i0} = 0, \quad T_{ij} = p\,\delta_{ij},$$

in units with $c = 1$. The energy-momentum tensor of a scalar field $\phi(t)$ with potential $V(\phi)$ is

$$T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - \frac{1}{2}\, g_{\mu\nu}\, \partial^\kappa \phi\, \partial_\kappa \phi + g_{\mu\nu}\, V(\phi).$$

Show that if spatial gradients can be ignored, the scalar field acts as a perfect fluid with energy density and pressure

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\,\dot{\phi}^2 + V(\phi), \\ p_\phi &= \frac{1}{2}\,\dot{\phi}^2 - V(\phi), \end{aligned}$$

where overdots denote differentiation with respect to time. [5]

Assume that the field is slowly rolling, $\dot{\phi}^2 \ll V(\phi)$. Use the equation-of-state parameter, $w = p/\rho$, to discuss the relevance of this result to,

(a) inflation in the early Universe,

(b) dark energy at late times.

[5]

(ii) The equation of motion in Planck units of a uniform scalar field, $\phi(t)$, in a Friedmann-Robertson-Walker universe with scale factor $R(t)$ is

$$\ddot{\phi} + 3\,H\,\dot{\phi} = -V',$$

where overdots denote differentiation with respect to time and $V' = \partial V/\partial\phi$. The Hubble parameter $H(t) = \dot{R}/R$ is given by

$$H^2 = \frac{1}{3}\,(\rho_{\rm m} + \rho_{\rm r} + \rho_\phi),$$

where $\rho_{\rm m}$, $\rho_{\rm r}$ and ρ_ϕ are the energy densities of dark matter, radiation and the scalar field ϕ. Using the expressions for p_ϕ and ρ_ϕ given in Part (i) or otherwise, express the parameter

$$x = \frac{\dot{\phi}^2}{2\,V},$$

in terms of the equation-of-state parameter $w_\phi = p_\phi/\rho_\phi$ of the scalar field. [3]

Use the equation of motion to show that

$$\xi \equiv \frac{1}{6}\,\frac{\mathrm{d}\ln x}{\mathrm{d}\ln R} = -\frac{V'}{V}\,\frac{\rho_\phi}{3\,\dot{\phi}\,H} - 1.$$

[10]

If the field is slowly evolving, so that $\xi \ll 1$, show that

$$\frac{3\,(1+w_\phi)}{\Omega_\phi} \approx \left(\frac{V'}{V}\right)^2,$$

where $\Omega_\phi = \rho_\phi/(3H^2)$ is the energy density of the scalar field in units of the critical energy density. [4]

It has been conjectured that in a consistent quantum theory of gravity $|V'/V| > 1$. If the dark energy is a slowly evolving scalar field, and $\Omega_\phi \approx 0.7$ at the present day, show that such a conjecture leads to a bound on the equation-of-state parameter w_ϕ. [2]

Is this bound consistent with observations? [1]

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) The luminosity of a star in the V-band can be approximated as

$$L_{\rm V} = 4\pi^2 R^2 \int_{\nu_{\rm V1}}^{\nu_{\rm V2}} B_\nu \, {\rm d}\nu,$$

where R is the star's radius, $\nu_{\rm V1}$, $\nu_{\rm V2}$ are the minimum and maximum frequencies of the V filter bandpass, $B_\nu(T) = (2h\nu^3/c^2)[\exp(h\nu/(k_{\rm B}T)) - 1]^{-1}$ is the Planck function, and $k_{\rm B}$ is the Boltzmann constant. Similar relations apply to the luminosity in the B-band and the U-band. Show that if a star has such a high effective temperature that $h\nu \ll k_{\rm B}T$ for the radiation in the U, B, and V bands, the $(B - V)$ colour can be approximated by

$$(B - V) \simeq -2.5 \log_{10} \left(\frac{\nu_{\rm B2}^3 - \nu_{\rm B1}^3}{\nu_{\rm V2}^3 - \nu_{\rm V1}^3} \right),$$

and is therefore a constant independent of the temperature T, and that the same applies to the $(U - B)$ colour. [4]

Interstellar dust absorbs and scatters radiation preferentially at shorter wavelengths. A star that is dimmed by E magnitudes in the V-band is dimmed by $1.3\,E$ magnitudes in the B-band and by $1.5\,E$ magnitudes in the U-band. How does interstellar extinction affect the position of a star in a $(U - B)$ vs $(B - V)$ diagram? [3]

Show that it is possible to define a parameter $Q = (U - B) - K(B - V)$, where K is a constant you should determine, such that Q is independent of the amount by which the star is affected by interstellar extinction. [3]

(ii) In a simple model of the internal structure of a spherical star, the distributions of temperature T and density ρ with radius are approximated by,

$$T = T_{\rm c} \exp(-r^2/a^2), \qquad \rho = \rho_{\rm c} \exp(-nr^2/a^2),$$

where r is the distance from the centre, $T_{\rm c}$ and $\rho_{\rm c}$ are the values of temperature and density at $r = 0$, and n and a are positive constants. The energy generation rate is assumed to be of the form $\epsilon = \epsilon_0 \rho T^\eta$, where η is a another positive constant. Let $L(r)$ be the luminosity within radius r. Show that the total luminosity of the star is

$$L(\infty) = \frac{\pi^{3/2} \epsilon_0 \, a^3 \, \rho_{\rm c}^2 \, T_{\rm c}^\eta}{(2n + \eta)^{3/2}}.$$

[5]

Let $M(r)$ be the total mass within radius r. Show that the inverse of the mass-to-light ratio of the star is

$$\frac{L(\infty)}{M(\infty)} = \epsilon_0 \, \rho_{\rm c} \, T_{\rm c}^{\eta} \left(\frac{n}{2n+\eta} \right)^{3/2} . \qquad [5]$$

Show that the ratio L/M decreases by a factor $[n/(2n+\eta)]^{3/2}$ from the centre to $r = \infty$. [5]

Assuming hydrostatic equilibrium, derive an approximation for the pressure profile $P(r)$ near the core, where $r \ll a/\sqrt{n}$, to leading order in nr^2/a^2. [5]

[Hint: you may use the definite integral: $I_n = \int_0^\infty x^n \, e^{-\alpha x^2} {\rm d}x = \frac{n-1}{2\alpha} \, I_{n-2}$ *and* $I_0 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$.*]*

TURN OVER...

Question 5Z - Statistical Physics

(i) What distinguishes bosons from fermions? [1]

Consider a gas of N non-interacting ultra-relativistic electrons in a large fixed 3-dimensional cubic volume V. Show that the density of states $g(E)$ at energy E is,

$$g(E) = \frac{V}{\pi^2 \hbar^3 c^3} E^2.$$ [5]

Using the grand partition function, show that $pV = A\langle E \rangle$, where p is the pressure, $\langle E \rangle$ is the average energy, and A is a numerical constant that you should determine. [4]

(ii) What are the implications of the properties of fermions for their occupation number of states and for their ground state at low temperature? [2]

For the gas of non-interacting ultra-relativistic electrons described in Part (i), show that the Fermi energy $E_{\mathrm{F}} = D\,(N/V)^{1/3}$, where D is a constant that you should determine. [5]

Show that at zero temperature $pV^a = K$, where a and K are constants that you should determine. You may express your answers in terms of the constant A in Part (i). How does this compare to an ultra-relativistic classical ideal gas? [5]

Now consider the same system with a magnetic field B, which changes the energy of an electron by $\pm \mu_B B$ depending on whether the spin is parallel or anti-parallel to the magnetic field, and μ_B is a constant. Assuming that $\mu_B B \ll E_{\mathrm{F}}$, show that, at zero temperature, the total magnetic moment is given by

$$M \approx \alpha\, \mu_B^{\gamma}\, B^{\delta}\, g(E_{\mathrm{F}}),$$

where $g(E_{\mathrm{F}})$ is the density of states at energy E_{F} and α, γ and δ are numerical constants that you should find. Then find the susceptibility, χ, of the gas at zero temperature. Comment on the result. [8]

Question 6Z - Principles of Quantum Mechanics

(i) Consider a hydrogen atom with $\{|n, \ell, m\rangle\}$ denoting the simultaneous eigenstates of the Hamiltonian H, and the angular momentum operators L^2 and L_z. Show that $[L_z, z] = 0$ and hence that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $m' = m$. [5]

Show that parity implies that this matrix element vanishes if $\ell' = \ell$. [5]

(ii) Consider the hydrogen atom as in Part (i). Given that $[L^2, [L^2, z]] = 2\hbar^2(L^2 z + z L^2)$, show that this relation implies that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $|\ell' - \ell| = 1$ or $\ell' = \ell = 0$. [8]

A hydrogen atom in its ground state $|n, \ell, m\rangle = |1, 0, 0\rangle$ is placed in a constant, uniform electric field $\mathbf{E}$. With reference to the atom's charge distribution, but without detailed calculation, give a physical explanation of why there is no correction of first-order in $\mathbf{E}$ to the ground state energy, but higher-order corrections are possible. [6]

Show that the second-order correction to energy of the ground state caused by the electric field is

$$\frac{e^2|\mathbf{E}|^2}{\mathcal{R}} \sum_{n=2}^{\infty} \frac{n^2}{1-n^2} \, |\langle n, 1, 0|z|1, 0, 0\rangle|^2 \, ,$$

where $-\mathcal{R}$ is the unperturbed energy of $|1, 0, 0\rangle$. [6]

[*You may assume that, when a Hamiltonian is perturbed by ΔH, the second-order correction to the ground state energy is*

$$\sum_{\alpha} \frac{|\langle \alpha|\Delta H|\phi\rangle|^2}{E_\phi - E_\alpha} \, ,$$

where $|\alpha\rangle$ are a complete set of unperturbed eigenstates states orthogonal to the unperturbed ground state $|\phi\rangle$, and E_α, E_ϕ are their unperturbed energies.]

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) An observer comoving with the Sun measures the Doppler shifts of clouds along a line of sight that makes an angle l with the vector pointing from the Sun to the Galactic Centre (that is the clouds along a given line of sight all have Galactic longitude l). Explain, making clear your assumptions, how it is possible to construct the Galactic rotation curve using lines of sight at longitudes in the range $l < 90°$ or $l > 270°$. [7]

What other information is needed if lines of sight with $90° < l < 270°$ are used instead? [2]

Explain why it is likely that the orbits of gas clouds are closer to being circular than the orbits of stars. [1]

(ii) A cylindrical coordinate system is located at the centre of a thin ring of mass m and radius a, with the z-axis being oriented perpendicular to the plane of the ring. Show that the potential generated by the ring at radius $R \ll a$ in the $z = 0$ plane can be approximated by

$$\Phi(R) \approx -\frac{Gm}{a}\left(1 + \frac{R^2}{4a^2}\right).$$

[10]

A test particle experiences the potential produced by such a ring as well as that due to a point mass, M, located at the origin. Determine the minimum value of M such that the particle can execute stable circular orbits (in the plane of the ring) at all radii interior to the ring. [10]

[*You may assume if necessary that the solution to Laplace's equation for an axisymmetric matter distribution can be written in the form*

$$\Phi = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-(n+1)})P_n(\cos\theta).$$

The first three Legendre polynomials are given by $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$.]

Question 8Y - Topics in Astrophysics

(i) Describe three processes by which stellar binaries can form in a cluster, as well as the role (if any) of tides in those processes. [3]

Consider a protoplanetary disc orbiting a star of mass $M_\star$ which has a solid mass surface density of Σ at a distance from the star of r. A fraction $\eta = 10\%$ of the disc is converted into single planetesimals of mass M_p. Derive the following constraint on the vertical thickness H of the planetesimal disc such that interactions amongst the planetesimals result in the formation of binary pairs,

$$H/r \lesssim \left[\eta \left(M_\mathrm{p}/M_\star\right)^2 \left(\frac{\Sigma}{M_\star/r^2}\right)\right]^{1/7}.$$

You may assume that relative velocities amongst the planetesimals are set by their vertical motion. [5]

Determine the above constraint for the following situations involving a Sun-like star and 10^{19} kg planetesimals: (a) terrestrial planet formation at $r = 1\,\mathrm{au}$ for which $\Sigma = 2\,M_\oplus\,\mathrm{au}^{-2}$, (b) Kuiper belt formation at $r = 40\,\mathrm{au}$ for which $\Sigma = 10^{-4}\,M_\oplus\,\mathrm{au}^{-2}$. [2]

(ii) A planetesimal of radius R_p and density ρ_p is orbiting a white dwarf of mass $M_\star$ and density $\rho_\star$. Interactions with an outer planetary system result in it being scattered onto an orbit with pericentre at a distance r_p from the white dwarf. Derive the critical disruption pericentre $r_{\mathrm{p,crit}}$ for the planetesimal to be tidally disrupted. [3]

A planetesimal reaches this critical disruption pericentre for the first time. Providing justification, sketch the shape of the planetesimal as it passes through pericentre, and describe the subsequent evolution of the planetesimal fragments and how it depends on their initial location within the planetesimal. [4]

To approximate the tidal disruption process, consider the planetesimal as it reaches pericentre, which is in the x-direction at $x = r_\mathrm{p}$ from the white dwarf, where $r_\mathrm{p} \gg R_\mathrm{p}$. Ignore any tidal perturbation up to this point, and so assume the planetesimal has remained spherical. Now imagine that the planetesimal's self-gravity and internal strength are turned off during pericentre passage. Derive the new semimajor axis of the portion of the planetesimal which, at the time of pericentre passage, was at a distance from the white dwarf of $x = r_\mathrm{p} + d$. You may assume that the planetesimal's orbital velocity v and separation from the star r are related as $0.5\,v^2 - GM_\star/r \approx -0.5\,GM_\star/a$, where a is the semimajor axis of its orbit. [4]

CONTINUE OVER...

Derive a constraint on the planetesimal's orbital eccentricity such that all of its fragments remain bound to the white dwarf following disruption. [2]

For the same assumptions for the tidal disruption process, now consider a planetesimal on an initially parabolic orbit. Show that the rate at which mass returns to the point of disruption as a function of time t is of the form $\mathrm{d}M/\mathrm{d}t = A\,t^{-5/3} + B\,t^{-3}$, where A and B are constants that should be determined. [5]

Comment on the subsequent evolution of the bound fragments. [2]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 10 June 2022 9am– 12pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.** *Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.*

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 7Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X - Relativity

(i) State Birkhoff's theorem. [2]

Assume spherical polar coordinates $x^\mu = (t, r, \theta, \phi)$ to describe a 4-dimensional spacetime. Consider a spherically symmetric non-rotating static thin shell of matter with mass M located at distance R from the origin. Use Birkhoff's theorem to derive a metric for the spacetime for $r < R$ and $r > R$. [4]

For which values of M and R is the shell hidden behind an event horizon. [2]

Comment on whether you would expect such a static shell of matter to be a stable solution of the Einstein field equations. [2]

(ii) A spaceship approaches a dark star on a radial geodesic and sends photons radially outwards as distress signals to a distant observer who is planning a rescue mission. The dark star is a static spherically symmetric object of mass M and radius R. Assume that the spaceship started its journey at infinity from rest. Consider the Lagrangian for particles orbiting outside a spherically symmetric mass distribution,

$$L = c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2(\theta)\dot{\phi}^2,$$

where $L = c^2$ for massive particles and $L = 0$ for massless particles. Overdots denote differentiation with respect to an affine parameter λ. Comment on the value of μ and use the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right),$$

to find the conserved quantities for the spaceship and the emitted photons. [6]

What is the energy of photons received by a stationary observer at radius r_{obs} in terms of the energy of the photons at infinity? [10]

What is the maximum redshift of a signal detected by a distant observer for which the spacecraft is at a radius r from which a rescue mission is possible? [4]

Question 2Z - Astrophysical Fluid Dynamics

(i) Explain the difference between Eulerian and Lagrangian perturbations, as well as why it is useful to consider Lagrangian perturbations when considering adiabatic fluctuations of a stratified system. [2]

Starting from the Lagrangian form of the continuity equation, show that the Lagrangian density perturbation $\Delta\rho$ and perturbation displacement $\boldsymbol{\xi}$ are related by $\Delta\rho + \rho_0 \, \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$, where ρ_0 is the unperturbed density. [3]

Proceed to show that the linearized continuity equation for Lagrangian perturbations about an equilibrium that is stratified in the z-direction is

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0, \qquad (*)$$

where Δu_z is the z-component of the corresponding velocity perturbations. [5]

[*You may assume that Eulerian perturbations* δX *and Lagrangian perturbations* ΔX *are related as* $\delta X = \Delta X - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla}) X_0$.]

(ii) Consider a local patch of a geometrically thin accretion disk that surrounds a central point mass M. In the comoving frame of that patch, the disk material is assumed to form an isothermal hydrostatic atmosphere with adiabatic sound speed c_{s}. Show that the density profile is

$$\rho_0 = A \exp\left[-\gamma \Omega^2 z^2 / (2 c_{\mathrm{s}}^2)\right],$$

where Ω is the angular frequency of the disk at that location, γ is the adiabatic index, z is the distance from the mid-plane, and A is the mid-plane density. [5]

Now suppose that this atmosphere is subjected to vertical, adiabatic perturbations. Starting from its standard form, show that the linearized momentum equation in the z-direction is

$$\frac{\partial \Delta u_z}{\partial t} = -\frac{c_{\mathrm{s}}^2}{\rho_0} \frac{\partial \Delta\rho}{\partial z} - \Omega^2 \xi_z,$$

where $\Delta\rho$ is the Lagrangian density perturbation, Δu_z is the vertical velocity, and ξ_z is the vertical displacement. [6]

CONTINUE OVER...

Show that density perturbations which have time dependence $\propto e^{-i\omega t}$ obey

$$\frac{\partial^2 \Delta\rho}{\partial \tilde{z}^2} + \tilde{z}\,\frac{\partial \Delta\rho}{\partial \tilde{z}} + \frac{1}{\gamma}\left(\frac{\omega^2}{\Omega^2} - 1\right)\Delta\rho = 0,$$

where $\tilde{z} = (\gamma\,\Omega^2/c_{\rm s}^2)^{1/2} z$. You may assume the result $(*)$ from Part (i) if necessary. [6]

The disk atmosphere can support normal modes for which $\Delta\rho \to 0$ as $z \to \infty$. Using the result above, show that there is a spectrum of such modes with frequencies $\omega = \Omega(1 + n\gamma)^{1/2}$, where n is a non-negative integer, and comment on the nature of the behaviour for $n = 0$. [3]

[*You may assume without proof that solutions $y(x)$ to the ordinary differential equation $y'' + xy' + ny = 0$ diverge as $y \sim e^{x^2/2}$, unless n is an integer in which case $y(x)$ decays as $y \sim e^{-x^2/2}$.*]

Question 3X - Introduction to Cosmology

(i) Consider perturbations of a Friedmann-Robertson-Walker universe dominated by a pressureless fluid with mean density $\overline{\rho}$. On scales $\lambda \ll ct$, the perturbations can be described by Newtonian theory leading to the non-linear equations,

$$\begin{aligned}
&\partial\delta/\partial t + \nabla \cdot (\mathbf{u}(1+\delta)) = 0, \\
&\partial\mathbf{u}/\partial t + 2\,(\dot{R}/R)\,\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -(1/R^2)\,\nabla\Phi, \qquad (*) \\
&\nabla^2\Phi = 4\pi\, G\,\overline{\rho}\,\delta\, R^2,
\end{aligned}$$

where $R(t)$ is the scale factor, $\delta = (\rho - \overline{\rho})/\overline{\rho}$ is the fractional overdensity, $\mathbf{u}(t) = \dot{\mathbf{x}}(t)$ is the comoving velocity and Φ is the gravitational potential. Spatial derivatives are with respect to the comoving coordinate $\mathbf{x}(t)$ and overdots denote differentiation with respect to time. If the perturbations are small, show that the perturbation equations $(*)$ give the linear perturbation equation,

$$\frac{\partial^2\delta}{\partial t^2} + 2\,\frac{\dot{R}}{R}\,\frac{\partial\delta}{\partial t} = 4\pi G\,\overline{\rho}\,\delta. \qquad (**)$$

[5]

Find the solutions to $(**)$ in a matter-dominated universe with zero curvature and $\Lambda = 0$ for which $R(t) \propto t^{2/3}$. [5]

(ii) Using the definitions and notation of Part (i), consider perturbations in a Friedmann-Robertson-Walker universe satisfying the equations $(*)$. Assume that the fluid flow is irrotational so that the comoving peculiar velocity can be written as $\mathbf{u} = -\dot{D}\,\nabla\psi$, where ψ is a time independent velocity potential and $D(t)$ satisfies the linear density perturbation equation $(**)$ of Part (i). Show that in linear perturbation theory, $\mathbf{u}$ is related to the gravitational acceleration $\nabla\Phi$ as

$$\mathbf{u} = -\frac{\dot{D}}{D}\,\frac{\nabla\Phi}{4\pi\, G\,\overline{\rho}\, R^2}.$$

[5]

Define $\mathbf{w} = \mathbf{u}/\dot{D}$. Show that the second non-linear equation in $(*)$ can be written as

$$\frac{\partial\mathbf{w}}{\partial D} + (\mathbf{w}\cdot\nabla)\,\mathbf{w} = \mathbf{S} = -\frac{1}{\dot{D}^2}\left[\frac{1}{R^2}\,\nabla\Phi + 4\pi\, G\,\overline{\rho}\,\frac{D}{\dot{D}}\,\mathbf{u}\right]. \qquad (***)$$

[10]

CONTINUE OVER...

In linear perturbation theory, the source term $\mathbf{S}$ in this equation is zero. Adopting $\mathbf{S} = 0$ as an approximate non-linear model of structure formation, show that $(***)$ requires that $\mathbf{w}$ remains constant along paths that follow the fluid flow. Hence show that the fluid elements travel along straight lines

$$\mathbf{x} = \mathbf{q} + D(t)\mathbf{w},$$

where the components of $\mathbf{q}$ are constants. [3]

Discuss briefly whether this model provides an accurate description of the non-linear evolution of structure in an expanding universe. [2]

Question 4Y - Structure and Evolution of Stars

(i) Describe the most important features of the optical spectra of white dwarfs, outlining the physical properties indicated by each spectral characteristic. [7]

Briefly outline the spectral classification of white dwarfs. [1]

Explain in a few sentences why astronomers are interested in discovering white dwarfs with low values of T_{eff}. [2]

(ii) A white dwarf star may be modelled as an isothermal degenerate core with temperature T_{c}, mass M_{c}, and molecular weight μ_{c}, which cools and loses energy at a rate

$$L = -\frac{3}{2}\frac{\mathcal{R}M_{\mathrm{c}}}{\mu_{\mathrm{c}}}\frac{\mathrm{d}T_{\mathrm{c}}}{\mathrm{d}t},$$

where $\mathcal{R}$ is the gas constant. This core is overlaid by a thin non-degenerate envelope where the opacity κ follows Kramers' law,

$$\kappa = \frac{A\rho}{T^{3.5}},$$

where ρ is the mass density and A is a constant. The transition density from core to envelope is given by $\rho_{\mathrm{t}} = CT_{\mathrm{c}}^{3/2}$, where C is a constant. Using equations of stellar structure for the envelope, show that pressure and density in the envelope are related by,

$$P \propto \rho^{\frac{n+1}{n}},$$

with $n = 3.25$. [8]

Show further that the luminosity depends on the core temperature as $L \propto T_{\mathrm{c}}^{3.5}$. [3]

Show that the luminosity decreases with time as $L \propto t^{-1.4}$. [3]

Explain how you would verify empirically that the time dependence of the luminosity derived above holds for real white dwarfs using a ground-based telescope. [6]

TURN OVER...

Question 5Y - Statistical Physics

(i) Consider a solid body with heat capacity at constant volume C_V. Assume that the solid's volume remains constant throughout Part (i) of this question. If the temperature changes from $T_{\rm i}$ to $T_{\rm f}$, show that the entropy change is $\Delta S = S_{\rm f} - S_{\rm i} = C_V \ln{(T_{\rm f}/T_{\rm i})}$. [2]

Now consider that two identical such bodies (both with heat capacity C_V) with initial temperatures T_1 and T_2 are brought into equilibrium in a reversible process. What are the final temperatures of the bodies? [4]

Now suppose that the two bodies are instead brought directly into thermal contact (irreversibly). What are the final temperatures of the bodies? Compute the entropy change and show that it is positive. [4]

(ii) State Carnot's theorem. [2]

Consider two Carnot engines. The first operates between two heat reservoirs at temperatures T_1 and $T_2 < T_1$. The second engine operates between two heat reservoirs at temperatures T_2 and $T_3 < T_2$. Show how Carnot's theorem can be used to define a thermodynamic temperature. [6]

Consider the Gibbs free energy, $G = E + pV - TS$, where E is energy, p is pressure, V is volume and S is entropy. Explain why $G = \mu(T,p)N$, where μ is the chemical potential and N is the number of particles. [3]

What is a first-order phase transition? [2]

Consider a system at constant pressure where phase I is stable for $T > T_0$ and phase II is stable for $T < T_0$. Explain how in a transition from phase II to phase I, $S_I - S_{II} > 0$, where S_I is the entropy in phase I and S_{II} is the entropy in phase II. [7]

[*Hint: Consider* $S = -\left(\frac{\partial G}{\partial T}\right)_{p,N}$ *for each phase.*]

Question 6Y - Principles of Quantum Mechanics

(i) A particle of mass m travels in one dimension subject to the Hamiltonian

$$H_0 = \frac{P^2}{2m} - U\,\delta(x)\,,$$

where P is the momentum operator, U is a positive constant and $\delta(x)$ is the Dirac delta function. Let $|0\rangle$ be the unique bound state of this potential and E_0 its energy. Further, let $|k,\pm\rangle$ be unbound H_0 eigenstates of even or odd parity, each with energy E_k, chosen so that $\langle k',+|k,+\rangle = \langle k',-|k,-\rangle = \delta(k'-k)$. At times $t \leq 0$ the particle is trapped in the potential well. From $t = 0$, it is disturbed by a time-dependent potential. The perturbed Hamiltonian is then,

$$H = H_0 + \lambda v(x,t),$$

where $v(x,t) = -Fx\,e^{-i\omega t}$ and $F > 0$ and $0 < \lambda \ll 1$ are constants. Subsequently, the particle's state may be expressed as

$$|\psi(t)\rangle = a(t)\,e^{-iE_0 t/\hbar}|0\rangle + \int_0^\infty \left(b_k(t)|k,+\rangle + c_k(t)|k,-\rangle\right)\,e^{-iE_k t/\hbar}\,\mathrm{d}k\,.$$

Show that

$$\dot{a}(t)\,e^{-iE_0 t/\hbar}|0\rangle + \int_0^\infty e^{-iE_k t/\hbar}\left(\dot{b}_k(t)|k,+\rangle + \dot{c}_k(t)|k,-\rangle\right)\,\mathrm{d}k = \frac{i\lambda F}{\hbar}e^{-i\omega t}\,x|\psi(t)\rangle$$

for all $t > 0$. [5]

Show that $b_k(t) = 0$, working to first order in λ. [5]

(ii) For the system in Part (i), and working to first order in λ, show that

$$c_k(t) = \frac{i\lambda F}{\hbar}\langle k,-|x|0\rangle\,e^{i\Omega_k t/2}\,\frac{\sin(\Omega_k t/2)}{\Omega_k/2}\,,$$

where $\Omega_k = (E_k - E_0 - \hbar\omega)/\hbar$. [8]

CONTINUE OVER...

The original bound state has position space wavefunction $\langle x|0\rangle = \sqrt{K}\, e^{-K|x|}$ where $K = mU/\hbar^2$, while the position space wavefunction of the odd parity unbound state is $\langle x|k,-\rangle = \sin(kx)\,/\sqrt{\pi}$ and its energy $E_k = \hbar^2 k^2/(2m)$. Show that, at late times, the probability that the particle escapes from the original potential well is

$$P_{\mathrm{free}}(t) = \frac{8\hbar\lambda^2 F^2 t}{mE_0^2}\frac{\sqrt{E_{\mathrm{f}}/|E_0|}}{(1+E_{\mathrm{f}}/|E_0|)^4},$$

to lowest order in λ, where $E_{\mathrm{f}} > 0$ is the final energy. [12]

[*You may assume that as $t \to \infty$, the function* $\sin^2(\eta t)/(\eta^2 t) \to \pi\,\delta(\eta)$.]

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Starting from the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0,$$

show that the first moment equation can be written in the form

$$\frac{\partial}{\partial t}(\nu \overline{v_j}) + \frac{\partial}{\partial x_i}(\nu \overline{v_i v_j}) + \nu \frac{\partial \Phi}{\partial x_j} = 0,$$

where you should define ν, $\overline{v_j}$ and $\overline{v_i v_j}$. [7]

If all the stars have mass m, write this equation in terms of the mass density, ρ, and identify the physical meaning of each term in this equation. [3]

(ii) The distribution function for a spherically symmetric system is described by

$$f = F\epsilon^{-1/2},$$

where ϵ is the relative energy and F is a normalisation constant. Explain (a) why this distribution function is a solution of the steady-state collisionless Boltzmann equation, and (b) why it describes a system where the velocity distribution is everywhere isotropic. [3]

Show that the stellar density ρ and the relative potential Ψ are related by $\rho \propto \Psi$. [3]

Derive the functional form for $\rho(r)$ and show that the outer radius of the cluster scales as $F^{-1/2}$. [5]

Show that the mean square stellar speed is proportional to ρ. [3]

Sketch the form of the probability density function for the stellar speed at a fixed location. [3]

Compare this with the corresponding probability density function for molecular speeds within the analogous fluid system and comment on why these distributions are not the same. [3]

[*You may assume that for a spherically symmetric potential, $\Phi(r)$, the Laplacian operator can be written*

$$\nabla^2 \Phi = \frac{1}{r}\frac{d^2(r\Phi)}{dr^2}. \text{]}$$

TURN OVER...

Question 8Z - Topics in Astrophysics

(i) A survey at a wavelength $\lambda = 1.1\,\mathrm{mm}$ covering an area of $A = 10$ square arcmin has detected a number of galaxies and measured their flux densities S (in mJy) down to a limit of $S_{\mathrm{lim}} = 100\,\mu\mathrm{Jy}$. These detections have been used to determine the number of galaxies per square degree with flux densities in the range S to S+dS to be

$$n(S)\mathrm{d}S = N_0(S/S_0)^{-\alpha}\mathrm{d}(S/S_0),$$

where $N_0 = 2700\,\mathrm{deg}^{-2}$, $S_0 = 2.6\,\mathrm{mJy}$, and $\alpha = 1.81$. Estimate how many galaxies were detected. [5]

Estimate the flux density of the brightest galaxy that is likely to have been detected in the survey. [2]

Comment on whether it is the brightest or faintest galaxies that contribute most to the cosmic infrared background at this wavelength. [3]

(ii) Consider a disc of carbon monoxide (CO) gas around a star of mass $M_\star$ at a distance d from the Sun. The gas orbits at radii in the range R_{in} to R_{out}. Observations of an optically thin CO emission line are made and the disc is found to be both spatially and spectrally resolved. The image of the disc shows its midplane to be edge-on to our line of sight. Construct a *position-velocity diagram* in which the x-axis is the projected distance from the star along the disc's midplane as seen on the sky, and the y-axis is the radial velocity of the gas relative to the star V_r. Sketch the region within which the gas emission must lie, quantifying the locus of any boundaries. [8]

Describe how such a position-velocity diagram can be used to determine the mass of the star, as well as the radial distribution of gas. [2]

The star undergoes a flare and it is suggested that the CO emission might respond to this event leading to variability in the position-velocity diagram. Quantify the time delay in the response expected as a function of V_r and x, and sketch where the perturbation to the position-velocity diagram would be expected at a time R_{in}/c after the flare is first detected for $R_{\mathrm{out}}/R_{\mathrm{in}} = 3$. [8]

Discuss some of the challenges with detecting the response to the flare in this way. [2]

END OF PAPER