OUTLINE PROOF – MAXIMUM LIKELIHOOD METHOD

The likelihood is the probability of observing a particular dataset, therefore

$$\int L(x \mid \theta)\ dx = 1$$

differentiate with respect to θ

$$\int \frac{\partial L}{\partial \theta}\ dx \ = \ 0 \ = \ \int \frac{1}{L}\ \frac{\partial L}{\partial \theta}\ .L\ dx \ = \ \int \frac{\partial ln(L)}{\partial \theta}\ .L\ dx$$

differentiate RH term with respect to θ again

$$\int \frac{\partial^2 ln(L)}{\partial \theta^2}\ .L + \left(\frac{\partial ln(L)}{\partial \theta}\right)^2 .L\ dx = 0$$

therefore

$$\left\langle\ -\ \frac{\partial^2 ln(L)}{\partial \theta^2}\ \right\rangle \ = \ \left\langle\ \left(\frac{\partial ln(L)}{\partial \theta}\right)^2\ \right\rangle$$

Let t be an unbiased estimator of some function of θ, say $\tau(\theta)$, then

$$<\ t\ > = \int t\ L\ dx \ = \ \tau(\theta)$$

$$\tau'(\theta) \ = \ \frac{\partial \tau(\theta)}{\partial \theta} \ = \ \int t\ \frac{\partial ln(L)}{\partial \theta}\ L\ dx$$

therefore from above

$$\tau'(\theta) \ = \ \int (t - \tau(\theta))\ \frac{\partial ln(L)}{\partial \theta}\ L\ dx$$

Use Schwarz inequality on $\tau'^{\ 2}$ to generate

$$\tau'^{\ 2} \ \leq \int (t - \tau)^2\ L\ dx \ \times \ \int \left(\frac{\partial ln(L)}{\partial \theta}\right)^2 L\ dx$$

Therefore, for the case $\tau(\theta) = \theta$

$$var\{t\} \geq \frac{1}{\left\langle\ \left(\frac{\partial ln(L)}{\partial \theta}\right)^2\ \right\rangle} = \frac{1}{\left\langle\ -\ \frac{\partial^2 ln(L)}{\partial \theta^2}\ \right\rangle}$$

MALMQUIST BIAS

(also known as Eddington bias)

...... or Eddington's solution of Fredholm's integral equation of 1st kind

$$F(x) \;=\; \int U(x-z)\; K(z)\; dz$$

Expand the LH integral argument to give

$$U(x-z) \;=\; U(x) \;-\; U'(x)\; z \;+\; U''(x)\; \frac{z^2}{2!} \;-\; ......$$

Integrate term by term

$$F(x) \;=\; \sum_n \frac{\mu_n}{n!}\; U^{(n)}(x)(-1)^n$$

where μ_n are moments of integration kernel K. Now rewrite

$$U(x) \;=\; F(x) + \sum_n A_n F^{(n)}(x)$$

For a central Kernel (ie. $\mu_1 = 0$) and equating coefficients

$$U(x) \;=\; F(x) - \frac{\mu_2}{2!}F^{(2)}(x) + \frac{\mu_3}{3!}F^{(3)}(x) - \left[\frac{\mu_4}{4!} - \left(\frac{\mu_2}{2!}\right)^2\right] F^{(4)}(x) + ...$$

For a Gaussian kernel $\mu_{odd} = 0$, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$, ... therefore

$$U(x) \;=\; F(x) - \frac{\sigma^2}{2}F^{(2)}(x) + \frac{\sigma^4}{8}F^{(4)}(x) + ...$$

and for say a luminosity function of the form eg. $N(m) = 10^{\alpha(m-m_o)}$

$$\frac{dN}{dm} \;=\; ln10\;\alpha\; N(m) \qquad\qquad \frac{d^2N}{dm^2} \;=\; (ln10)^2\;\alpha^2\; N(m)$$

$$N_{obs}(m) \;=\; N(m) + \frac{\sigma^2}{2}\;(ln10)^2\;\alpha^2\; N(m) \quad \text{equivalently} \quad \Delta m \;=\; -ln10\;\alpha\;\frac{\sigma^2}{2}$$