N-Body Simulations

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Direct Methods

Close Encounters

Open Clusters

Post-Newtonian Dynamics

GPU Implementation

Newton's Equations

Force $\mathbf{F}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$

Explicit differentiation

$$\mathbf{F}_i^{(1)} = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

New solution at $t = \Delta t$

$$\Delta \dot{\mathbf{r}}_i = (\frac{1}{2}\mathbf{F}_i^{(1)}\Delta t + \mathbf{F}_i)\Delta t$$

$$\Delta \mathbf{r}_i = ((\frac{1}{6}\mathbf{F}_i^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_i)\Delta t + \dot{\mathbf{r}}_i)\Delta t$$

Basic Integration

Taylor series $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_0^{(2)}\Delta t^2 + ...$

Fourth-order scheme   Explicit $\mathbf{F}$ and $\mathbf{F}^{(1)}$

Corrector for i

$$\begin{aligned} \Delta \mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\Delta t^5 \\ \Delta \dot{\mathbf{r}}_i &= \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\Delta t^4 \end{aligned}$$

Convergent time-step $\Delta t = \left(\frac{\eta|\mathbf{F}|}{|\mathbf{F}^{(2)}|}\right)^{1/2}$

Relative criterion Δt independent of mass

Block-steps $\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$

Scheduling $i = \min\,(t_j + \Delta t_j)$

Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n << \Delta t_d, \qquad n << N$$

Neighbour sphere $R_s^{\text{new}} = R_s^{\text{old}} \left(\frac{n_p}{n}\right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

Close Encounters

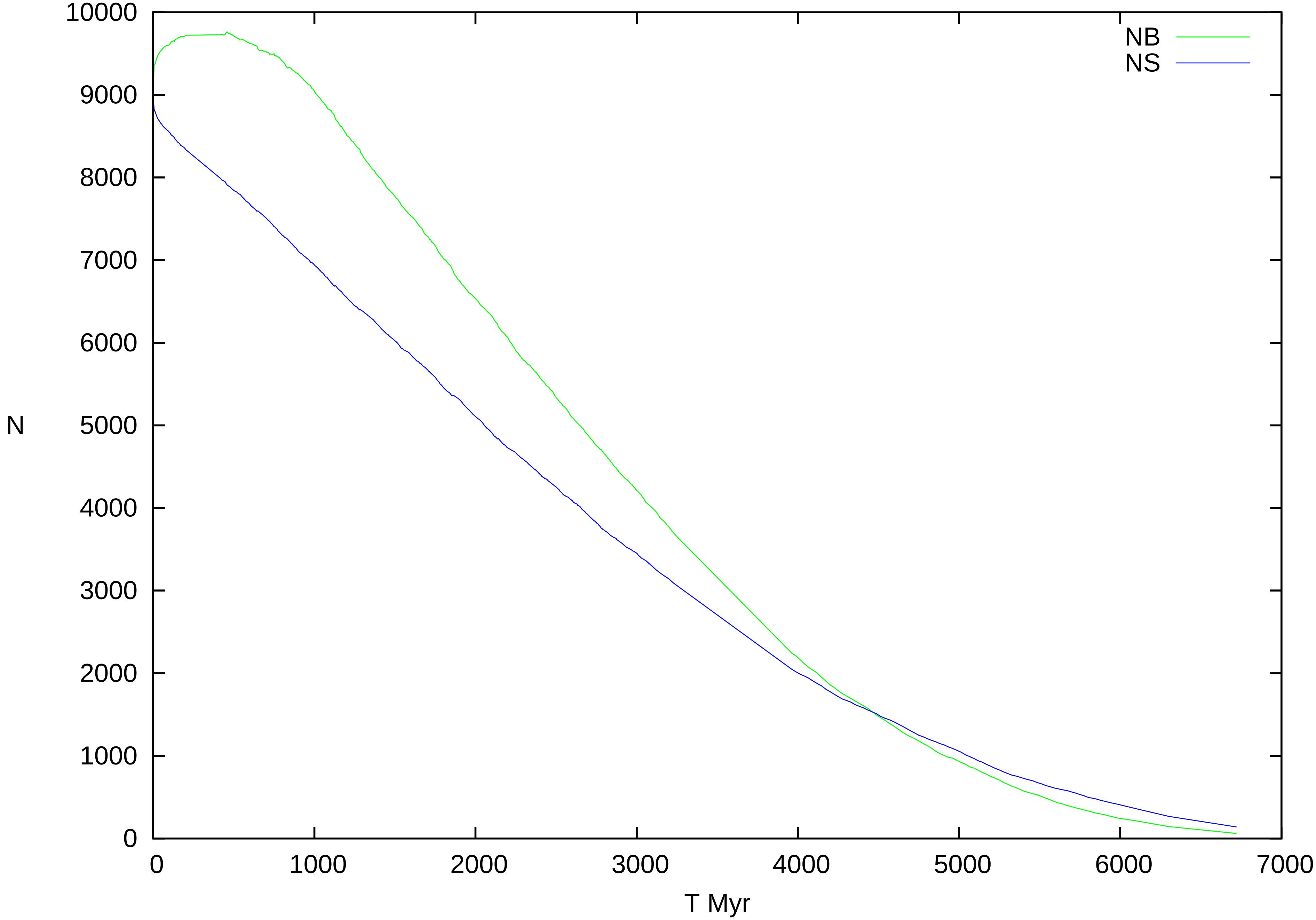
Binaries	Two-body regularization (1965)
Triples	Three-body regularization (1972)
Multiples	Chain regularization (1989)
Hierarchies	Stability criterion (2008)
Aim	Improved treatment & efficiency
Cost	Programming complexity
Advantage	Two-body description

Hierarchical Systems

Hierarchical formation	B + B $\Rightarrow$ T + S or B + $\tilde{\mathrm{B}}$, $e_{\rm out} < 1$
Dynamical molecules	[B,S], [B,B], [[B,S],S], [[B,B],S]
Formation rate	binary–binary (Mikkola 1982)
Stability	general criterion (Mardling 2008)
Chaos boundary	fuzzy region & holes
Inclination effect	prograde vs retrograde stability
Kozai cycles	$\cos^2 i\,(1 - e_{\rm in}^2) = \mathrm{const}$
Higher-order systems	quartet, quintuplet, sextuplet
Instability	$\dot{e}_{\rm out} > 0 \;\Rightarrow$ slingshot

Binary Processes

Tidal circularization	$a\,(1 - e^2) = \text{const} \quad \Rightarrow \quad \dot{a} < 0$
Roche-lobe mass transfer	$r^* > r_{\rm RL}, \quad \Delta\, m_2 = -f \Delta\, m_1$
Common envelope evolution	$m_c > 0, \quad \text{MS} + \text{giant}$
Magnetic braking	$\dot{a}_{\rm MB} \propto a^{-4}$
Gravitational radiation	$\dot{a}_{\rm GR} \propto a^{-3}$
Spin-orbit coupling	$J_{\rm tot} = J_{\rm orb} + J_{\rm spin}$
Stellar collisions	$a\,(1 - e) < 0.75\,(r_1^* + r_2^*)$
Blue stragglers	mass transfer or MS collisions
Cataclysmic variables	WD + giant
X-ray objects	WD + MS, NS + MS
Doubly degenerates	WD + WD, $P \simeq 10$ mins
Type Ia supernova	WD – WD collision or inspiral



NB
NS
N
T Myr
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

Post-Newtonian Terms

Equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

First-order precession

$$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$

$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$

$$B_1 = 2(2-\eta)\dot{r}$$

Higher-order precession

$$A_2 = ..., \quad B_2 = ..., \quad A_3 = ..., \quad B_3 = ...$$

Gravitational radiation

$$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$

$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation

$$\mathbf{P}_{GR} = \frac{M}{c^2 r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

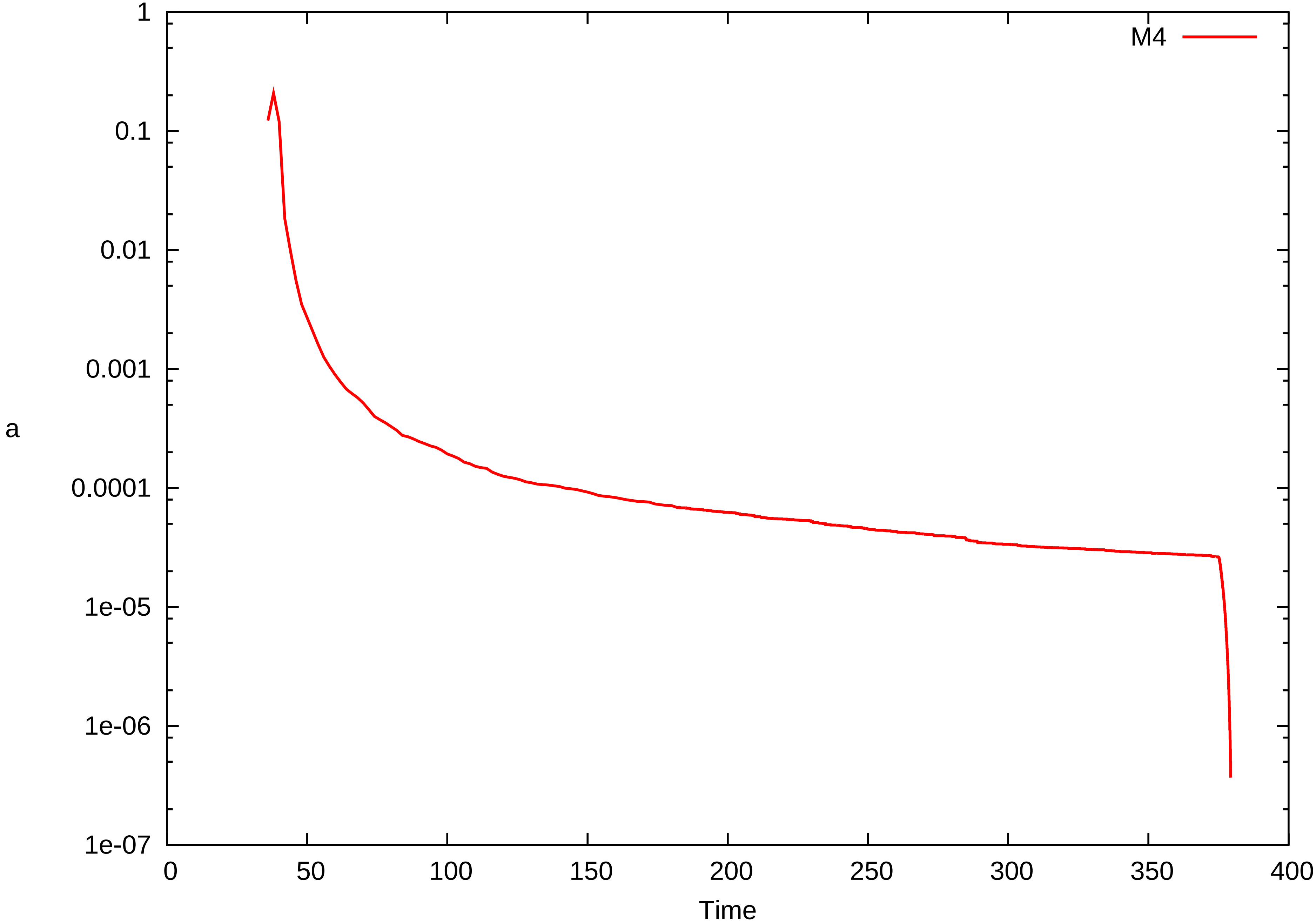
Radiation energy loss

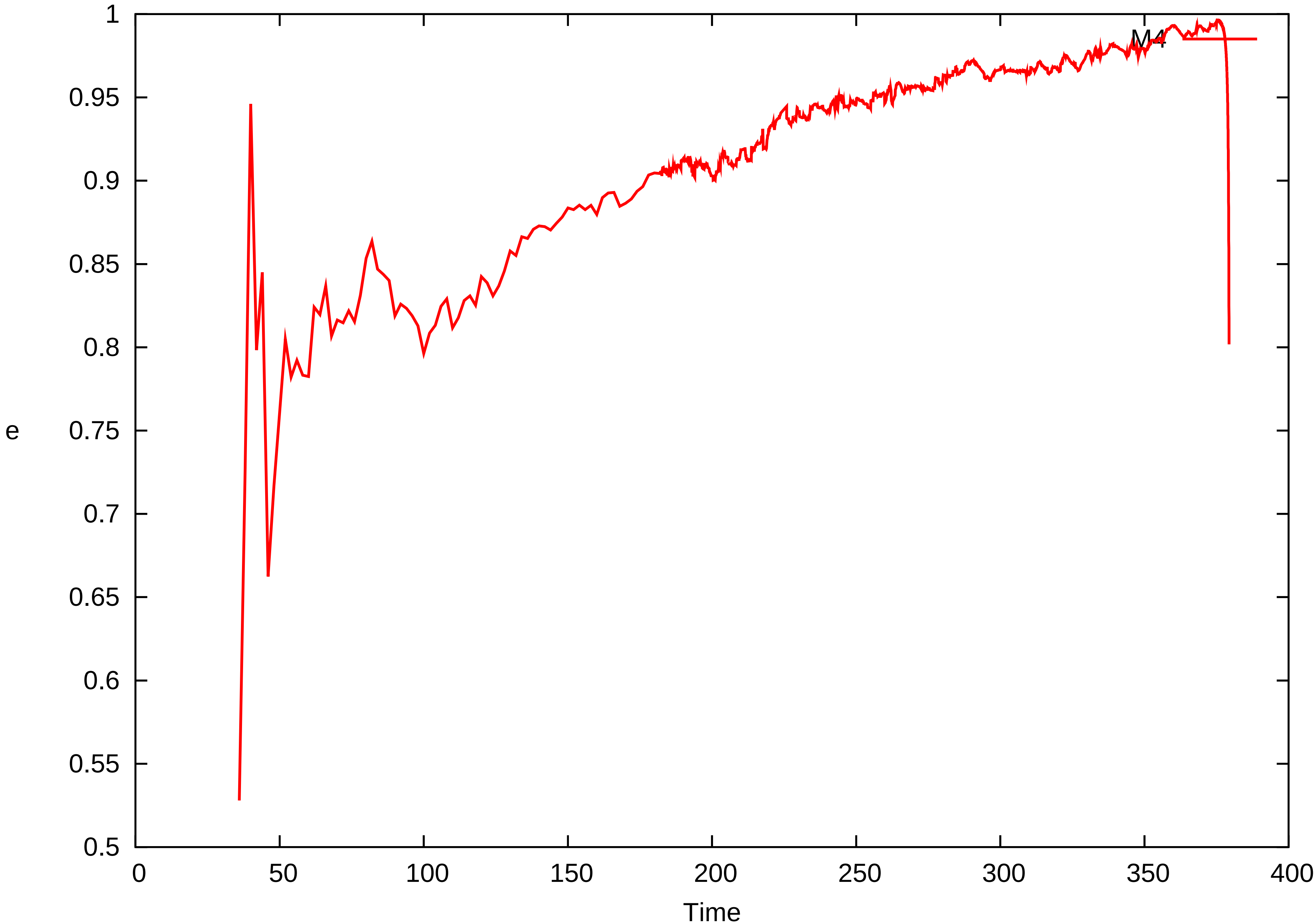
$$\Delta E_{GR} = \frac{m_1 m_2}{M}\int \mathbf{P}_{GR}\cdot\mathbf{v}\,dt$$

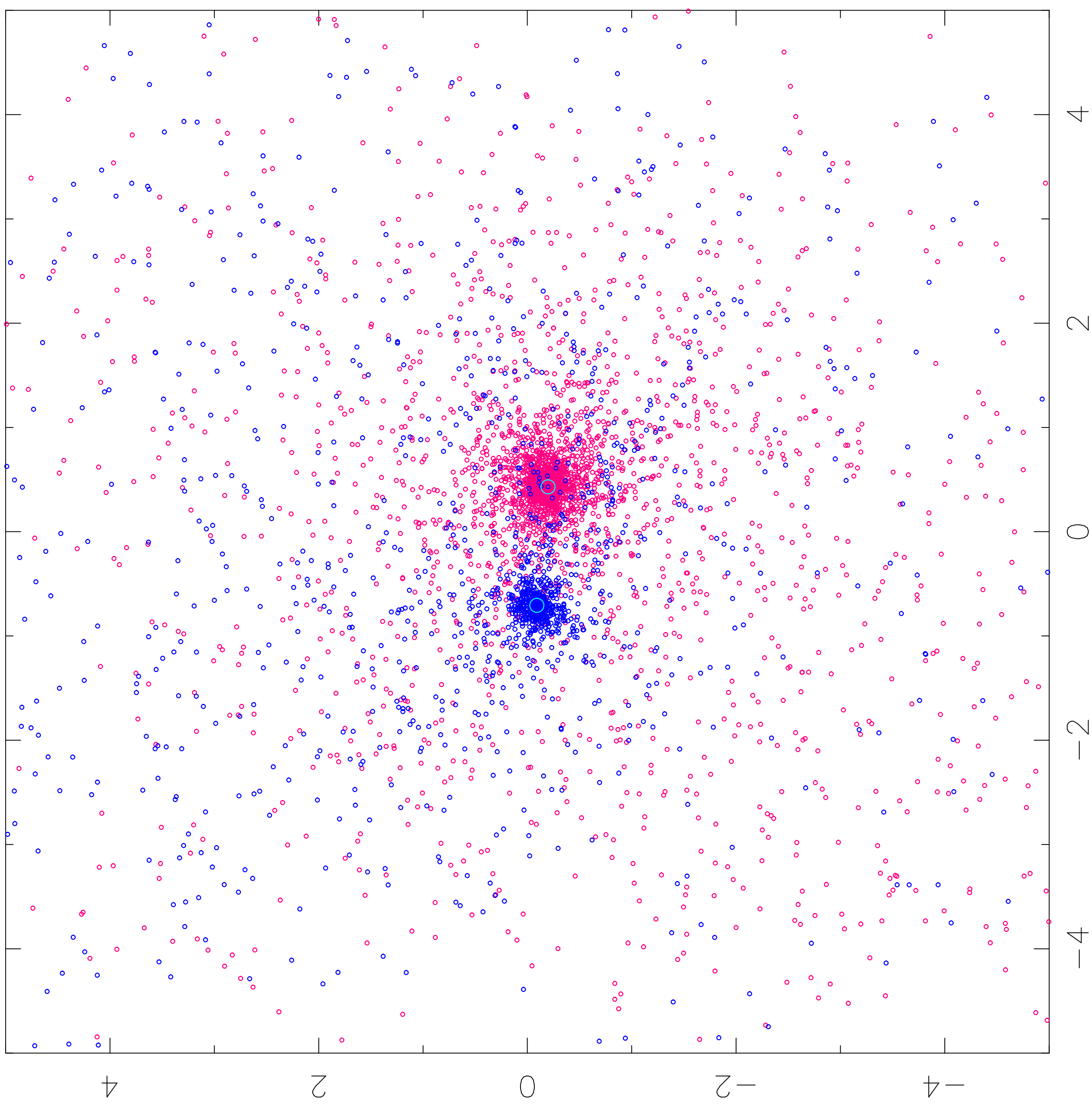
GR radiation time-scale

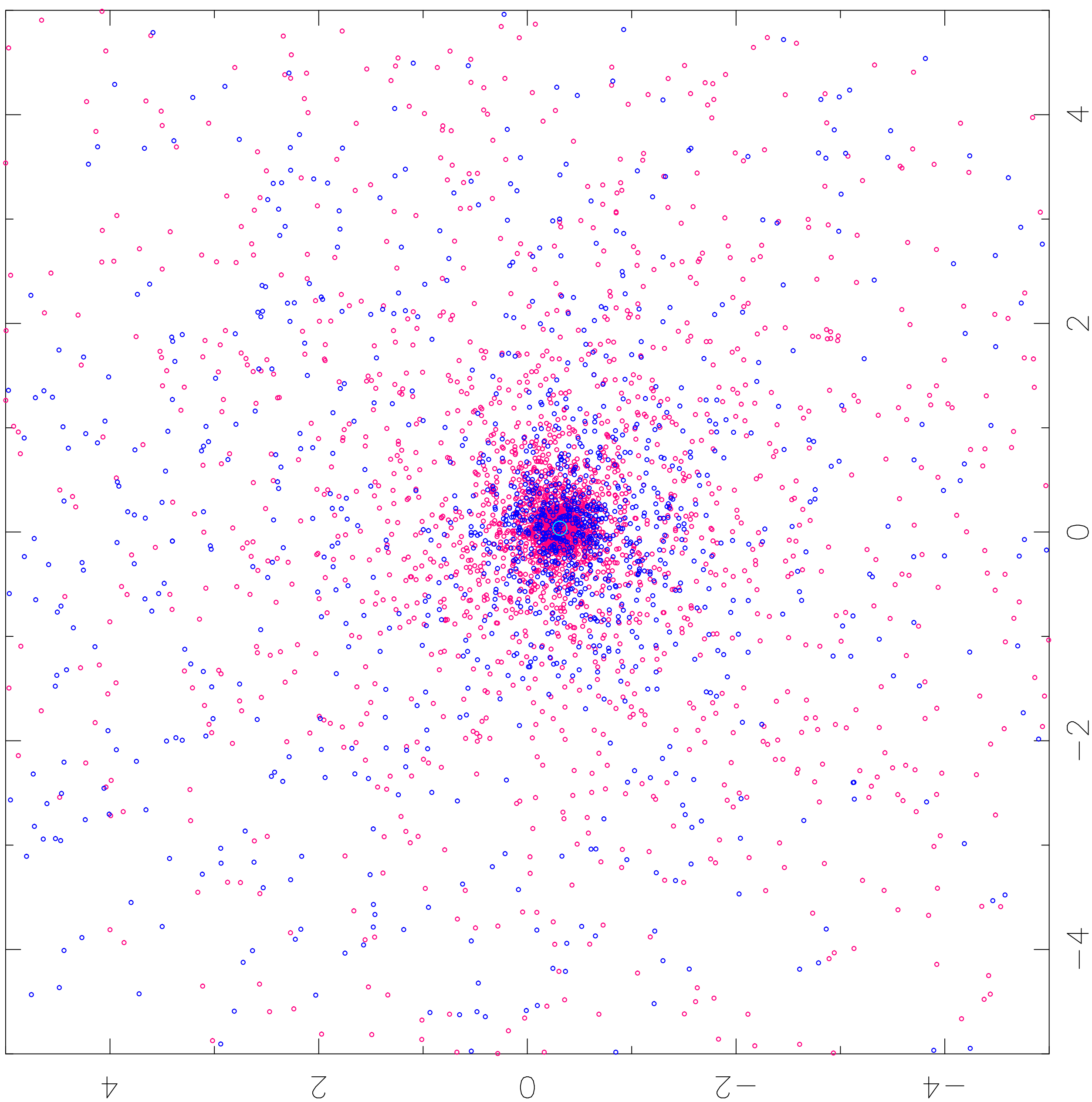
$$t_{\mathrm{GR}} = \frac{5}{64}\frac{c^5\, g(e)\, a^4}{X(1+X)\, m_x^3}, \quad c = \frac{3\times 10^5}{V^*}$$

Decision-making — graduated PN terms from v/c or t_{GR}


M4
a
Time
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
0
50
100
150
200
250
300
350
400


e
Time
M4
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
50
100
150
200
250
300
350
400





GPU System

Standard PC	Quad CPU
Gaming card	NVIDIA
Programming	CUDA library
Implementation	Parallel force calc
Code	NBODY6 + GPU
Speed-up	Factor of 30 for $250
Performance	Exceeds GRAPE-6A
Timing	$N = 64\text{K}$, CPU $= 10\,\text{mins}/T_{\text{cr}}$

Colour-Magnitude Diagram Legend:

- single main-sequence (MS) star, MS-MS binary
- single white dwarf (WD)
- WD-WD binary
- MS-WD binary [active CV]
- MS star in binary (non-MS or WD companion)
- Blue Straggler (BS)
- sub-giant, giant, or supergiant star
- naked Helium star
- WD in binary (non-MS or WD companion)
- Neutron star or Black Hole (only shown if in binary)

e.g. BS-WD binary

Upper-Right Panel:

Cumulative radial profiles of selected sub-populations (at current time):

- single MS stars
- MS-MS binaries
- single giants
- single WDs

Lower-Right Panel:

Evolution of selected cluster properties to the current time:

- number density of stars in the core
- cluster mass as fraction of initial cluster mass (scales from 1 to 0)