Problem Set I: G. Efstathiou, Particle Astrophysics

1. FRW model, relativity and gravitational lensing

(1.1) Show that the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mathrm{mu}\nu} = -8\pi G T_{\mu\nu},$$

can be rewritten as

$$R_{\mu\nu} - \Lambda g_{\mathrm{mu}\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\mu}_{\mu} \right).$$

(1.2) Derive expressions for the luminosity and angular diameter distance in a spatially flat matter dominated FRW model with $\Lambda = 0$. Show that the angular diameter of a standard rod is a minimum at redshift $z = 5/4$. Explain physically why the angular diameter increases at high redshift. Repeat the calculation for the de-Sitter solution, $R(t) \propto \exp(Ht)$.

(1.3) Use conformal time τ (defined by $d\tau = dt/a$) to show that the Friedman equations ($\Lambda = 0$) can be rewritten as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1, \qquad 2\frac{d\mathcal{H}}{d\tau} = -(3w+1)(\mathcal{H}^2 + kc^2),$$

where $\mathcal{H} = R^{-1} dR/d\tau$, w (assumed constant) is the equation of state parameter $P = w\rho c^2$, Ω is the density parameter $\rho/\rho_{\mathrm{crit}}$, and $\rho_{\mathrm{crit}} = 3\mathcal{H}^2/(8\pi G R^2)$. Hence derive the evolution equation for Ω given in the lectures:

$$\frac{d\Omega}{d\tau} = (3w+1)\mathcal{H}\Omega(\Omega - 1).$$

For both $w = 0$ and $w = -1$, sketch the evolution of Ω as a function of τ in an expanding universe for the two cases $\Omega > 1$ and $\Omega < 1$ at early times.

(1.4) The Lagrangian of a freely-falling particle is

$$\mathcal{L} = [g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}]^{1/2}, \qquad \dot{x}^{\alpha} = \frac{dx^{\alpha}}{dp},$$

and satisfies the Euler-Lagrange equation

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial x^{\alpha}} = 0,$$

where p is a parameter that varies along the timelike path of the orbit. Show that

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}^2}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}^2}{\partial x^\alpha} = 2\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\frac{d\mathcal{L}}{dp}. \tag{1}$$

If p is an affine parameter, show that the right hand side of this equation is zero, so we recover the usual form of the geodesic equations of motion,

$$\ddot{x}^\nu + \Gamma^\nu_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0.$$

(1.5) Using the dimensionless quantites in the lecture notes, show that lensing of a 'small' source by a point mass object leads to magnifications of the two images of

$$\mu_\pm = \frac{1}{4}\left(\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} \pm 2\right).$$

What does 'small' mean in this context?

(1.6) The energy-momentum tensor of an infinite straight cosmic string lying along the z-axis of a cylindrical coordinate system (r, θ, z) is

$$T_{\mu\nu} = \rho(r)c^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \rho(r) = \begin{cases} \rho_0 & r \le r_0 \\ 0 & r > r_0 \end{cases}.$$

Give a physical interpretation of this tensor, and deduce the tension of the cosmic string.

Show that the metric

$$ds^2 = c^2dt^2 - dr^2 - f^2(r)d\theta^2 - dz^2,$$

is a solution of the Einstein field equations if

$$\frac{1}{f(r)}\frac{d^2f(r)}{dr^2} = -\frac{8\pi G}{c^2}\rho(r).$$

and show that

$$f(r) = \begin{cases} \frac{1}{k}\sin kr & r \le r_0 \\ a + br & r > r_0 \end{cases},$$

where k, a and b are constants.

If $kr_0 \ll 1$, show that at large distances from the string ($r \gg r_0$)

$$f^2(r) \approx r^2(1 - 8G\mu/c^2),$$

where $\mu = \pi\rho_0 r_o^2$ is the mass per-unit-length of the string. Give a physical interpretation of this result and discuss how gravitational lensing could be used to detect cosmic strings.

2. Thermal History

(2.1) Show that for particles in thermal equilibrium at temperature T, the comoving entropy

$$S = \frac{R^3(t)}{T}(\rho + P),$$

is conserved.

(2.2) The distribution function for massless neutrinos of type i is

$$f_i(p) = \frac{1}{\left[\exp\left(\frac{(pc-\mu_i)}{kT_\nu}\right) + 1\right]},$$

where T_ν us the neutrino temperature and μ_i is the chemical potential. Show that in the degenerate limit $|\mu_i| \gg kT_\nu$, the number density of each neutrino type is

$$n_{\nu_i} \approx \frac{4}{3}\frac{\pi}{h^3c^3}|\mu_i|^3.$$

How does μ_i depend on the scale factor $R(t)$? Show that the contribution of degenerate massless neutrinos to the mean mass density of the Universe is

$$\rho_{\nu_i} \approx \frac{\pi|\mu_i|^4}{h^3c^5}.$$

Is degeneracy compatible with upper limits to the mean mass density of the Universe? Discuss how neutrino degeneracy would affect the production of primordial helium.

(2.3) The number density of particles of mass m in equilibrium in the early Universe is given by the integral

$$n = \frac{4\pi g_s}{h^3}\int_0^\infty \frac{p^2 dp}{\exp[(E(p)-\mu)/kT] \pm 1}, \qquad \begin{cases} + \text{ fermions,} \\ - \text{ bosons,} \end{cases}$$

where $E(p) = c\sqrt{p^2 + m^2c^2}$, μ is the chemical potential, and g_s is the spin degeneracy. Give a physical interpretation of the chemical potential μ.

What can you say about the chemical potentials of photons, particles and their antiparticles?

Assuming that $\mu \ll mc^2$ and that particles remain in equilibrium when they become non-relativistic, show that their number density can be approximated as

$$n \approx g_s \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \mathrm{e}^{(\mu - mc^2)/kT}.$$

(2.4) At around $t = 100$ seconds, deuterium D forms through the nuclear fusion of non-relativistic protons p and neutrons n via the interaction $p + n \leftrightarrow D$. In equilibrium, what is the relationship between the chemical potentials of the three species? Show that the ratio of their number densities can be expressed as

$$\frac{n_D}{n_n n_p} \approx \left(\frac{\pi m_p kT}{h^2}\right)^{-3/2} \mathrm{e}^{B_D/kT},$$

where the deuterium binding energy is $B_D = m_n + m_p - m_D$ and you may take $g_D = 4$.

Now consider the fractional densities $X_a = n_a/n_B$, where n_B is the baryon number of the Universe. Relate the ratio $X_D/(X_n X_p)$ to the baryon-to-photon ratio, η, of the Universe.

Why does deuterium form only at temperatures much lower than given by $kT \approx B_D$?

(2.5) The main reaction responsible for maintaining hydrogen and radiation in thermal equilibrium is

$$p + e \leftrightarrow H + \gamma.$$

Assume that the particle densities are given by the non-relativistic Maxwell-Boltzmann formula

$$n_i = g_i \left(\frac{2\pi m_i c^2 kT}{h^2}\right)^{3/2} \exp\left(\frac{\mu_i - m_i c^2}{kT}\right),$$

where g_i, μ_i and m_i are respectively the number of spin states, chemical potential and mass of particle i. Show that

$$\frac{n_e n_p}{n_H} \approx \left(\frac{2\pi m_e c^2 kT}{h^2}\right)^{3/2} \exp\left(-\frac{B_H}{kT}\right),$$

where $B_H = 13.6$ eV $= (158000K)/k$ is the binding energy of hydrogen.

Show that the ionization fraction $X_e = n_e/(n_e + n_H)$ is approximately,

$$\frac{X_e^2}{1 - X_e} \approx \left((1 - Y_{\text{He}}) \frac{\rho_B}{m_p} \right)^{-1} \left(\frac{2\pi m_e c^2 kT}{h^2} \right)^{3/2} \exp\left(-\frac{B_H}{kT} \right),$$

where ρ_B is the mean baryon density and Y_{He} is the helium abundance by mass. Explain why hydrogen recombination occurs at a significantly lower temperature than B_H/k.